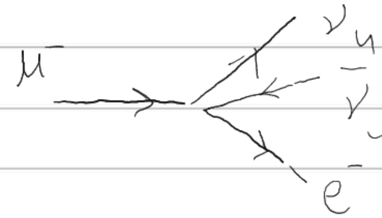
Since these are lectures on QFT & SM let me start with the first calculation most of us encounter in this area: the muon lifetime

$$\mu^- \to e^- \bar{\nu}_e \nu_\mu$$



$$\mathcal{L} = \frac{G_F}{\sqrt{2}} J^{(+)\alpha}_{(\mu)} \bar{J}^{(-)}_{(e)\alpha}$$

$$J^{(+)\alpha}_{(\mu)} = \bar{\nu}_\mu(x)\gamma^\alpha(1-\gamma_5)\mu(x)$$

Where $\mu(x)$ is a field that "annihilates" a muon (and creates an antimuon) and similarly for $\bar{\nu}_{(\mu)}$ (and of course the corresponding statements for $J^{(-)}_{(e)\alpha}$)

Evaluating the amplitude in lowest order perturbation theory, straightforwardly summing and averaging over spins, and integrating over phase space

$$\boxed{\Gamma = \frac{m_\mu^5 G_F^2}{192\pi^3}}$$

This calculation raises some obvious questions

- What do we mean "create" and "annihilate"?
- Why is this the right Lagrangian?
  - If not why do we use it?

Relativity plus Quantum Mechanics

QM: observables are linear operators that act on a Hilbert space

* what space? what operators?

Without relativity ... position is straightforward

ops: $\{p, q\}$ + all polynomials

Hilbert: position or momentum eigenstates as basis.

With Relativity: we already expect we need more than just position eigenstates —

if we try to localize a particle to a region whose size is small compared to the compton wavelength we need momenta $\gg m$. But this means we need states with energy $\gg m$. Since mass is not conserved in relativity we expect multi particle states to enter

(NB We might be wrong — but if so we will only some extra ink by including them)

So for the Hilbert space we will take the full space of all possible multiparticle states

"Fock space"

$|0\rangle; \; |\vec{k}\rangle, \; |\vec{k}_1 \vec{k}_2\rangle \; |\vec{k}_1, \ldots, \vec{k}_N\rangle; \ldots$

with the appropriate inner product $\langle \vec{k}_1 | \vec{k}_2 \rangle = (2\pi)^3 2\omega_{k_1} \delta^3(\;)$

$\langle k_1', k_2' | k_1, k_2 \rangle = (2\pi)^3 2\omega_{k_1} 2\omega_{k_2} \{ \delta^3(k_1 - k_1') \delta^3(k_2 - k_2') + \delta\delta \}$

and

$(H, \vec{P}) \equiv \hat{P}^\mu$ are sums over all particles

$$\hat{P}^\mu |\vec{k}_1 \ldots \vec{k}_N\rangle = \sum_j^N k_j^\mu \, |\vec{k}_1 \ldots \vec{k}_N\rangle$$

$$k^\mu \equiv (\omega_k, \vec{k}) \qquad \omega_k = \sqrt{\vec{k}^2 + m^2}$$

So much for the Hilbert space*
(i.e. completion of space with these states as basis)

What about observables? First notation —
Introduce operators $a^\dagger(k)$, all k, defined by

$$a^\dagger(\vec{q})\,|\vec{k}_1, \ldots, \vec{k}_N\rangle = |\vec{q}; \vec{k}_1 \ldots \vec{k}_N\rangle$$

From the inner product we deduce that, $a(\vec{q})$ removes a particle of momentum $\vec{q}$ from the state (or gives zero if no such particle is present); equivalently this gives

$$[a(k'), a^\dagger(k)] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}')$$

(and $[a^\dagger, a^\dagger] = [a, a] = 0$)

It is not hard to show that every linear operator on $\mathcal{H}$ can be constructed from $\hat{a}$ and $\hat{a}^\dagger$.

But what should our observables look like? I will impose a few conditions in constructing them the first is key — note that in relativity the causal structure is key — "EVENTS" (points in spacetime) may be causally disconnected (if they cannot communicate with signals propogating at speeds less than or equal to c)

∴ Observables must be labeled by their spacetime locations!
↳ $\vec{x}$ is "demoted" (on a par with t) from operator to label
There is no "position" to be measured

On to the requirements

1) $\phi(x) = \phi^\dagger(x)$

2) $\hat{P}^\mu$ should generate spacetime translations

$$e^{-i\hat{P}\cdot a}\phi(x)e^{i\hat{P}\cdot a} = \phi(x-a)$$

(NB ↳ $-i[\hat{H},\phi] = \dot{\phi}$)

3) $\hat{U}(\Lambda)^\dagger \phi(x) U(\Lambda) = \phi(\Lambda^{-1}x)$

4) $[\phi(x), \phi(y)] = 0 \qquad (x-y)^2 < 0$

5) linear in $a, a^\dagger$ ✱

$$\boxed{\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[e^{-ik\cdot x}a(k) + e^{ik\cdot x}a^\dagger(k)\right]}$$

Now we may easily verify

1) $\phi = \phi^{\dagger}$

2) $(\partial_0^2 - \vec{\nabla}^2 + m^2)\phi \equiv (\Box + m^2)\phi = 0$ Klein-Gordon

3) $[\dot{\phi}(t,\vec{x}), \phi(t,\vec{y})] = -i\delta^3(\vec{x}-\vec{y})$ CCR

4) $\phi(x)$ Scalar under Lorentz

But its not hard to show that this works in reverse also

FOCK SPACE + LOCAL OBSERVABLES $\longleftrightarrow$ CANONICAL QUANTIZATION of SCALAR FIELD.

In fact its not hard to see that this works for <u>ANY</u> Fock Space (including particles with spin)

$\therefore$ whats special about local fields is that they give rise to FOCK SPACE and can ensure Relativistic invariance (and Cluster Decomposition)

* What about the other direction? Counter Example string Theory.

Believe it or not, we can already get some physics mileage out of this discussion. I want to start in an odd place for lectures on QFT & SM, and ask a classic (19th century) physics question:

Why is the sky blue?

This was first answered (correctly) by Lord Rayleigh in 1871, when he used EM theory to calculate the scattering of EM waves from small dielectric spheres. This is now a (graduate) text-book exercise, but we can do it more simply with QFT.

1) The process is $\gamma + N_2 \to \gamma + N_2$

The molecule is non-relativistic —

We will need to develop a QFT to describe this process. The photon is easy, but the N_2 molecule is a bound state of N atoms, which are each a bound states of electrons and a nucleus, and each nucleus is a bound state of nucleons, each of which is a bound state of quarks and gluons, and...

Oh. Do we need to know what quarks and gluons are made of?

How can Fermi's theory accurately account for β decay processes? Don't we need the SM?
When solving for the energy levels of the Hydrogen atom we don't worry about the structure of the nucleus:

$$\text{nuclear size} \sim 10^{-15}\,m \times \text{few}$$
$$\text{atomic size} \sim 10^{-10}\,m \times 1/2 \quad \text{Bohr}$$

$\therefore$ The electron is spread out over a large region and the wave function varies slowly. In momentum space, the momenta of the electrons are smaller than those needed to probe the nucleus

$$P_{atom} \ll P_{nucleus} \approx \frac{1}{r_{nucleus}}$$

This simple observation, that measurements probing large distances are insensitive to short distance, is at the heart of understanding QFT & the SM
Back to the sky.

$$\lambda_{visible} \sim \text{few } 10^{-7}\,m \gg a_B$$

$\therefore$ The details of the molecule should be irrelevant
↳ introduce a field to annihilate the molecule (and create an antimolecule) $\phi(x)$

Hamiltonian? Non-relativistic $E = \vec{p}^2/2M$

$$H = \int d^3x\, \phi^\dagger\left(-\frac{\vec{\nabla}^2}{2M}\right)\phi = \int d^3x\, \frac{|\vec{\nabla}\phi|^2}{2M}$$

($*$ L must be $L = \int d^3x \left[\phi^\dagger i\partial_0 \phi \cancel{c\phi^\dagger} - \frac{|\vec{\nabla}\phi|^2}{2M}\right]$)

What is the dimension of ϕ? (mass)

$$[H] = 1 = \left[\int d^3x \frac{|\vec{\nabla}\phi|^2}{2M}\right] = -3+2-1+2[\phi]$$

$\hookrightarrow \quad [\phi] = 3/2$

Photons? $\quad H = \frac{1}{2}\int d^3x \, (\vec{E}^2 + \vec{B}^2) \qquad [\vec{E}] = [\vec{B}] = 2$

And now the interaction.

What should we choose? Clearly an ∞ number of choices! Lets make something up —

Annihilate incoming molecule $\rightarrow \phi$
create outgoing " $\rightarrow \phi^\dagger$
create & annihilate photons $\rightarrow \vec{E}^2, \vec{B}^2, \vec{E}\cdot\vec{B}$. rotational invariance

($\vec{E}\cdot\vec{B}$ parity odd; magnetic effects usually smaller for non-rel.)

$$H_I = \int d^3x \, \vec{E}^2 \phi^\dagger\phi \; e^2 \; a^3 \cdot C$$

e^2: Each $\vec{E}$ comes with e; a^3: (length)^3; C: dimensionless

An elementary calculation (Feynman diagrams) gives the Rayleigh scattering cross section

$$\frac{d\sigma}{d\Omega} \propto \frac{|C|^2 \alpha^2 a^6}{\lambda^4} \quad |\vec{\epsilon}^{(1)}\cdot\vec{\epsilon}^{(2)}|^2$$

$|\vec{\epsilon}^{(1)}\cdot\vec{\epsilon}^{(2)}|^2 \downarrow \{1, \cos^2\theta\}$

Famously $1/\lambda^4$ dependence means shorter wavelengths scatter more, hence the sky overhead is blue relative to the sky at the Horizon when the sun is low.

But why did this work? What about other ops?
↳ since $[\vec{E}]=[\vec{B}]=2$, $[\phi]=[\phi^\dagger]=1$
any more fields would have been suppressed by more powers of $\left(\frac{a}{\lambda}\right) \ll 1$.

Derivatives also have dimension $[\nabla \partial_\mu]=1$.

∴ At long wavelength the interaction is well-approximated by the lowest-dimension operator with the right symmetries. Corrections from other operators are small, but may be included in a systematic way.

Indeed this general approach can be turned into a rigorous procedure, generally going by the name "Effective Field Theory". The basic ideas grew out of understanding the physics of the mesons and other hadrons in the 1960s, especially in work by Weinberg, and eventually in the ~~late~~ 1970s and 80s turned into a full fledged formalism through the work of Ken Wilson (Nobel Prize).

Leaving out the details, Wilson explained:

Start with arbitrary physics with a cutoff Λ

$$S = \int d^4x \sum_i \mathcal{O}_i \, c_i(\Lambda)$$

(Although I have assumed the theory is local mostly to ensure Lorentz Invariance and Cluster Decomposition, this can be relaxed)

If we are interested in physics at a scale $p \ll \Lambda$, we may remove ("integrate out") all the modes with $\bar{\Lambda} \ll p \leq \Lambda$. This is a field theory with a cutoff $\bar{\Lambda}$ and

$$S = \int d^4x \sum_i \mathcal{O}_i \; c_i(\bar{\Lambda})$$

The operators $\mathcal{O}_i$ have dimension $[\mathcal{O}_i] = d_i$ then by dimensional analysis:

$$\boxed{c_i(\bar{\Lambda}) = c_i(\Lambda)\left(\frac{\bar{\Lambda}}{\Lambda}\right)^{4-d_i}}$$

(This gets modified slightly by renormalization — instead of $4-d_i$ it becomes δ_i which can be non-integral!; at weak coupling $\delta_i \simeq 4-d_i$)

Therefore at long distance $\bar{\Lambda} \ll \Lambda$, the only operators we need consider are those for which $\boxed{\delta_i < 0}$

since adding more fields or more derivatives generally increases δ_i, this is usually a finite set. For weakly coupled theories this is all ops of $4-d_i \leq 0$

→ $\boxed{\text{Operators of dim 4 or less}}$

In the distant past we called theories with only operators of dim 4 or less "Renormalizable". It was thought that such theories were special and it was necessary to only consider these special theories. In fact models of Massive Gauge Bosons were routinely excluded until 't Hooft and Veltman showed they were renormalizable.

But Wilson changed all this — Renormalizability is not an input, a property to be demanded of a sensible theory. It is an output: All sensible theories are well described at long distance by a renormalizable Lagrangian. In this way of speaking 't Hooft and Veltman showed that Spon. Broken gauge theories could arise as the long distance effective description of QFT.

Of course symmetries may restrict the operators that may appear —

For example we used rotational invariance to preclude operators linear in $\vec{E}$. More complicated symmetries may appear in the construction of the effective long distance Lagrangian.

We can now say more concretely what the SM is — it is the most general linear combination of operators of dimension 4 or less that may be constructed out of a certain set of fields consistent with a certain set of symmetries.

<u>Ex</u> Estimate the lifetime of the 2p state of Hydrogen

(Ans: $1.6\ 10^{-9}$s from expt.
or $\tau \simeq \left(\frac{3}{2}\right)^8 \frac{1}{\alpha^5 m_e}$)

Global symmetries

What is a symmetry? Consider some mechanical system with degrees of freedom $\{\phi\}$. Values of $\{\phi\}$ and their first derivatives at one time constitute a set of Initial Value Data (IVD). Time evolution produces Final Value Data (FVD)

$$\text{IVD} \xrightarrow{t} \text{FVD}$$

Now imagine a transformation from one set of IVD to another $T(\text{IVD}) = \text{IVD}'$

A symmetry is a transformation that gives $\text{FVD}' = T(\text{FVD})$. That is, the transformation commutes with time evolution

$$\begin{tikzcd} \text{IVD} \arrow[r, "t"] \arrow[d, "T"'] & \text{FVD} \arrow[d, "T"] \\ \text{IVD}' \arrow[r, "t"'] & \text{FVD}' \end{tikzcd}$$

This will be the case if the EOM are invariant under the transformation. Since the EOM are the variation of the action, this implies the transformation of the action is a surface term; equivalently the transformation of the Lagrangian is a total derivative

$$\delta \mathcal{L} = \partial_\mu F^\mu$$

There is a famous and useful relation between a ~~continuous~~ symmetry depending on a continuous parameter and a conserved current

Let a continuous symmetry be a transform

$$\phi(x) \longrightarrow \phi(x;\lambda)$$

and define $D\phi \equiv \lim_{\lambda\to 0} \dfrac{\phi(x;\lambda)-\phi(x)}{\lambda}$

~~Then [illegible] for~~ By assumption $D\mathcal{L} = \partial_\mu F^\mu$

Let the Action by $\int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi) \equiv S$

The EoM are $\delta S = \int d^4x \left(\dfrac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\mu \delta\phi + \dfrac{\delta \mathcal{L}}{\delta\phi}\delta\phi \right)$

$$= \int d^4x \underbrace{\left(-\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} + \frac{\delta \mathcal{L}}{\delta \phi} \right)}_{=0} \delta\phi + \text{surface}$$

Now define a current $J^\mu \equiv \dfrac{\delta \mathcal{L}}{\delta \partial_\mu \phi} D\phi - F^\mu$

then $\partial_\mu J^\mu = \partial_\mu \dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} D\phi + \dfrac{\delta \mathcal{L}}{\delta \partial_\mu\phi} \partial_\mu D\phi - \partial_\mu F^\mu$

$$= \underbrace{\left(\frac{\delta \mathcal{L}}{\delta \phi} D\phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\mu D\phi \right)}_{D\mathcal{L}} - D\mathcal{L}$$

$= 0$ ✓

More complicated symmetries

Typically our theories will have more than one transformation — but there are restrictions on what a set of transformations can be.
Clearly if T_1 and T_2 are symmetry transformations, so is their product.
Also clearly such a transform can be undone, so $T_{1,2}^{-1}$ are also symmetry transforms. Finally the identity is obviously allowed. These 3 defining properties are what mathematicians call a "group". They are well studied systems and have a more or less complete classification.
We have encountered one: $SO(3,1)$. You are familiar with another: $SU(2)$ for ~~avg~~ angular momentum and spin.
We may characterize a cont group by infinitesimal transformations — these give the Generators of the transformation (Ex translations with generator $\vec{p}$ $T = e^{i\vec{p}\cdot\vec{a}}$)

In general two symmetry transformations won't commute (Not quantum). Then

$$[T^a, T^b] = i\, c^{abc}\, T^c \qquad \text{"Algebra"}$$

e.g. $SU(2)$ $a=1,2,3$ and $c^{abc} = \epsilon^{abc}$

The groups that we are mostly interested in are $SU(N)$ — the algebra is determined by "Special" ($\det = 1$) "Unitary" ($U^\dagger U$) matrices $N \times N$.

The Representations of these groups are what we are after: the set of objects that mix up under a transformation. Our fields are then divided into sets $\{\phi_m^{a}\}$

such that $T^a(\phi_m) = t^a_{mn}\,\phi_n$

For some matrices t^a_{mn} satisfying the algebra.

Ex $SU(2)$ the "spin" $1/2$ rep. $\phi_1, \phi_2 \rightarrow$

$$T^a \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sigma^a \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$[\sigma^a, \sigma^b] = i\epsilon^{abc}\sigma^c \checkmark$$

(Some group theory asides — for all groups there are some small representations that can be used to make all the others — for $SU(2)$ we can take the doublet (spin $1/2$) and construct higher spin by taking products; for $SU(3)$ we can take the two different 3-dim reps, called the 3 and the $\bar{3}$ and do the same. These are called "Fundamental" reps)

Symmetries, Gauge Invariance, and All That

Our conclusion at the end of lecture 1 was simple — ~~include~~ identify the fields that create and annihilate the particles of the SM, and construct the most general Lagrangian out of operators of dimension 4 or less that respect appropriate symmetries. Today we will be more concrete about these symmetries.

The first and most important constraint is Lorentz Invariance, which is exquisitely well tested (deviations are less than parts in 10^{18}!)

There six "generators" of Lorentz transforms: 3 rotations and 3 boosts. The Algebra of these transformations is easily constructed

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$

$$[J^i, K^j] = i\epsilon^{ijk} K^k$$

$$[K^i, K^j] = -i\epsilon^{ijk} J^k$$

This called $SO(3,1)$

Our job is to construct Representations of this algebra: objects that mix up under Lorentz transforms

In fact there are 2 distinct kinds of objects we need

1) States with particles.
these must form Unitary Representations of the Group.

Why Unitary? Probabilities shouldn't change under a symmetry

$$|\langle x|\psi\rangle|^2 = |\langle x|U^\dagger(\Lambda)\,U(\Lambda)|\psi\rangle|^2$$

Wigner $\rightarrow$ $U^\dagger U = 1$.

2) Fields that create and annihilate particles in the Fock space
$\rightarrow$ finite dim rep (so a finite # of fields)
Don't care about Unitary since we don't require $\psi^\dagger\psi = \psi^\dagger\Lambda^\dagger\Lambda\psi$

1) States: labeled by momentum.
2 different cases.

$p^2 \equiv m^2 > 0 \rightarrow$ the rep is also labeled by spin: half-integer.

$p^2 = 0 \rightarrow$ the rep is labeled by "helicity" $\rightarrow$ component of spin in the direction of motion

NB 1) Helicity is frame independent only for massless particles
2) Lorentz transforms don't change helicity → $\pm\lambda$ only because of reality (CPT)

Finite-d Reps of SO(3,1).
Simple Trick: $\vec{J}^{(\pm)} \equiv \frac{\vec{J} \pm i\vec{K}}{2}$

then we have $SU(2) \times SU(2)$
Reps of SU(2): Each irreducible rep. labeled by a half integer s with $(2s+1)$ basis vectors.

↳ SO(3,1) labeled by 2 ½ integers (s_+, s_-) of dimension $(2s_+ + 1)(2s_- + 1)$

Ex
$(0,0)$ Scalar
$(\frac{1}{2},0)$ 2-component Weyl spinor ("L")
$(0,\frac{1}{2})$ " " " " ("R")
$(\frac{1}{2},\frac{1}{2})$ 4-vector
$(1,0) \oplus (0,1)$ EM Stress Tensor ($\vec{E}$ and $\vec{B}$)

NB $(s_+, s_-)^* = (s_-, s_+)$

so if we have a field ψ_L in $(\frac{1}{2},0)$; ψ_L^* in $(0,\frac{1}{2})$

What about a Dirac Spinor (4-components)

$(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ Reducible!

(that is, Lorentz transforms don't mix all 4 states)

Why you should avoid 4-component spinors

1) Its reducible: there is no reason (in general) for the properties of the two irreducible parts to be related

In some cases there may be other symmetries that relate them, and in such cases there may be a small advantag in using 4-component notation. But this is not the case for the SM! Theories in which you don't have such symmetries are called "chiral"

2) Since we want a Hermitian Hamiltonian if we have a ψ in $(\frac{1}{2},0)$ then we will always have ψ^* in $(0,\frac{1}{2})$. Wei are therefore free to choose ALL fields as $(\frac{1}{2},0)$ (and we will)

Whether a fermion is $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ is called "chirality" ("L" or "R")

For a massless particle, the field ψ_L will annihilate particles of helicity $-1/2$.

This means using 2-component spinor notation: Pauli Matrices Rather than γ^μ

(NB $\frac{1\pm\gamma_5}{2}$ are chirality projectors — they are useful for converting between 2 and 4 component notation)

Ex. $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$

so $\psi^\dagger \otimes \psi$ is always a 4-vector

$\psi^\dagger \bar{\sigma}^\mu \psi \qquad \bar{\sigma}^\mu = (1,-\vec{\sigma})$

~~$\psi\psi$~~ $\psi\otimes\psi \qquad (\frac{1}{2},0)\otimes(\frac{1}{2},0) = (1,0)\oplus\underset{\sim}{(0,0)}$

$(\psi^T i\sigma_2 \psi)$ is the ~~singlet~~ scalar

$\psi_+\psi_- - \psi_-\psi_+$ ~~that term~~ vanishes unless $\{\psi_-,\psi_+\}=0$.

Ex U(1) symmetry and charged scalar field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - V[|\phi|] \quad ; \quad \phi \to e^{i\lambda}\phi$$

$$D\phi = i\phi \quad \text{and} \quad D\phi^* = -i\phi^*$$

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*} = \partial^\mu\phi \quad \text{und} \quad \partial_\mu(\partial^\mu\phi) = -\frac{\delta V}{\delta\phi^*}$$

Easy to see $D\mathcal{L} = 0$ (always for Global symmetries)

$$J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}D\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}D\phi$$

$$= \partial^\mu\phi^* i\phi - \phi^* i\partial^\mu\phi$$

Consequences in the Quantum Theory: Ward Identities

$$G^\mu(z;x_1\ldots x_n) \equiv \langle 0|T\{J^\mu(z)\,\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\}|0\rangle$$

$$= \int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot z}\,\tilde{G}^\mu(k;x_1\ldots x_n)$$

Then
$$\partial_\mu G^\mu = \langle 0|T\{\underbrace{\partial_\mu J^\mu}_{0}\,\mathcal{O}_1\ldots\mathcal{O}_n\}|0\rangle$$

$$+\sum_{j=1}^{n}\langle 0|T\{\mathcal{O}_1\ldots[J^0(\vec{z},t),\mathcal{O}_j(\vec{x}_j,t)]\,\delta(t-t_j)\ldots\mathcal{O}_n\}|0\rangle$$

We may always choose $\mathcal{O}_j$ with definite charge:

$$[J^0(\vec{z},t),\mathcal{O}_j(\vec{x}_j,t)] = -iq_j\,\delta^3(\vec{z}-\vec{x}_j)\,\mathcal{O}_j$$

$$\rightarrow \boxed{\partial_\mu G^\mu = -i\sum_{j=1}^{n}q_j\,\delta^4(x_j-z)\,\langle 0|T\{\mathcal{O}_1\ldots\mathcal{O}_n\}|0\rangle}$$

Fourier transform: $\partial_\mu \to -ik_\mu$.

so $$k_\mu \tilde{G}^\mu = \sum_{j=1}^{n} q_j e^{ik\cdot x_j} \langle 0|T\{O_1 \ldots O_n\}|0\rangle$$

Now consider the limit $k_\mu \to 0$. Since the LHS has an explicit k_μ we expect $\to 0$

this says $\langle 0|T\{O_1 \ldots O_n\}|0\rangle = 0$ whenever $\sum_j q_j \neq 0$

$\therefore$ Charge Conservation!
We call this "Wigner-Weyl"

But there is a less obvious possibility:
what if there is some set of $\{O_j\}$ such that $\sum_j q_j \neq 0$ and $\langle 0|T\ldots|0\rangle \neq 0$?

We say that some charged operator has a "vacuum expectation value": the vacuum is not invariant under the symmetry. This requires

$$\lim_{k_\mu \to 0} k_\mu G^\mu \neq 0 \to G^\mu \text{ has a } \underline{\text{POLE}} \text{ at } k^2 = 0$$

This means the current and some charged operators couple to a <u>massless</u> particle:

Goldstone Theorem $\to$ Nambu-Goldstone Boson coupling to the current like k^μ/k^2

Roughly $J^\mu \sim \partial^\mu \theta$ so $\partial_\mu J^\mu = \Box\theta = 0$ ✓

Continuous Symmetry ⟶ Charge Conservation OR NGB.

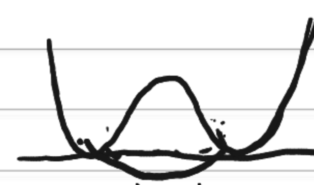
Lets quickly look at the simplest example:
A charged scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \frac{\lambda}{2}\left[|\phi|^2 - v^2\right]^2$$

Clearly the classical "vacuum" configuration is constant ϕ with $|\phi| = v$, that is

$$\phi = e^{i\lambda} v \quad \text{for some phase } e^{i\lambda}.$$

All of these configurations are related by a charge rotation, so if we understand the physics in one, we understand them all

The potential is a "mexican hat" 

with the minima forming a circle around the brim.

In QM there would be matrix elements between the degenerate states with $\langle\phi\rangle = v e^{i\lambda}$ and we would get a band of states with energies that vary. The minimum would be uniformly spread out over the trough.
However in ∞ volume these matrix elements

vanish and we have <u>SSB</u>: we must make a choice for which vacuum phase, e.g. $\theta = 0$
But we can consider long wavelength fluctuations about this point. Let

$$\phi = \frac{(v+\rho(x))}{\sqrt{2}} e^{i\theta(x)}$$ (a good parameterization except near $\rho = -v$)

Then

$$\mathcal{L} = |\partial_\mu \rho + i\partial_\mu \theta (\rho+v)|^2 - V[\rho+v]$$

$$= \frac{(\partial\rho)^2}{2} + (\rho+v)^2(\partial\theta)^2 - \frac{V''(v)}{2}\rho^2 + \cdots$$

① ρ is a massive field

② θ is massless — the NGB.

③ The current is $J^\mu = i\,\partial_\mu \phi^* \phi + c.c.$

$\simeq v^2 \partial^\mu \theta + \cdots$

and current conservation

$$\partial_\mu J^\mu \simeq v^2 \Box\theta = 0 \rightarrow \boxed{\Box\theta = 0}$$

Ex. A global symmetry $SU(2)\times U(1)$

A complex scalar field

	$SU(2)$	$U(1)$
	2	1/2

$$H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

That is, under $U(1)$ $\quad H \to e^{i\lambda/2} H$

under $SU(2)$ $\quad H \to e^{i\vec{\lambda}\cdot\vec{\sigma}/2} H$

Potential $\quad U[H] = \frac{\lambda}{2}\left[H^\dagger H - v^2\right]^2$

$$= \frac{\lambda}{2}\left[\phi^{+*}\phi^+ + \phi^{0*}\phi^0 - v^2\right]^2$$

$\cancel{\phi^+ = \phi_1 + i\phi_2}$ $\qquad \phi^0 = \phi_3 + i\phi_4$

~~Minimum~~ $\cancel{\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2} = v^2$

Minimum $\quad H^\dagger H = v^2$. Can always choose

$\mathrm{Re}\,\phi^0 = v$; all others zero:

$$\langle H\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

But this doesn't break all the symmetries:

$Q \equiv T^3 + Y \quad$ where $\quad QH = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} H$

is a residual symmetry.

$$SU(2)\times U(1) \to U(1)$$

GB?

$$H = e^{i\frac{\vec{\sigma}}{2}\cdot\vec{\Theta} + \frac{i}{2}\Theta_0}\begin{pmatrix} 0 \\ v+\rho \end{pmatrix}$$

Looks like 5 modes - however

$$\frac{1}{2}\begin{pmatrix} \sigma^3\Theta_3 + \Theta_0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Theta_3 + \Theta_0 & \\ & \Theta_0 - \Theta_3 \end{pmatrix}$$

and $\Theta_3 + \Theta_0$ doesn't appear in this expression. So we don't need to keep it (could choose ~~$\Theta_0 = \Theta_3$ or~~ $\Theta_0 = 0$ for example)

$$\mathcal{L} \stackrel{\sim}{=} (0, v)\,\partial_\mu e^{-i\frac{\vec{\sigma}}{2}\cdot\vec{\Theta}}\,\partial^\mu e^{i\frac{\vec{\sigma}\cdot\vec{\Theta}}{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\Sigma = e^{i\vec{\sigma}\cdot\vec{\Theta}/2} \to \mathcal{L} \simeq v^2\,\partial_\mu\Sigma^\dagger\partial^\mu\Sigma$$

"Non-linear Sigma Model"

Gauge Theory

We want to include fields for E&M
the quanta are PHOTONS that come in 2 polarizations: helicity ± 1 ("L" and "R" circularly polarized). This is consistent with our massless state representation (NB Lorentz transforms don't mix $+1$ with -1)

But there is no representation for fields on our list that is 2-dimensional (except $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ which are clearly not right)

How do we make a theory of photons?

You all know the answer (from E&M)
use a 4-vector potential $(\frac{1}{2},\frac{1}{2})$ A_μ.
But this has 2 problems
1) 4 is not equal to 2
2) $A_\mu A^\mu = A_0^2 - \vec{A}^2 \longrightarrow A_0$ will create states with the opposite norm to those created by A_i (i.e. negative probability)

These 2 famous problems have a famous slns

∴ Ensure the action does not involve all 4 fields. But we want to do this in a way that maintains Lorentz covariance

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

To see that this doesn't involve all the fields in A_μ

1) $F_{\mu\nu}$ doesn't involve $\dot{A}_0 \to$ no equation of motion for A_0, just an IV constraint (no propagating DOF)

2) There is gauge invariance

$$A_\mu \to A_\mu + \partial_\mu \Lambda \quad \text{for arbitrary fn } \Lambda$$

$$F_{\mu\nu} \to F_{\mu\nu}$$

$\therefore$ There is a part of A_μ (the "scalar" part) that doesn't enter the action.

This is NOT a symmetry — it does not relate sets of IVD. It arises since we are using an overcomplete parameterization of physical operators (to preserve manifest Lorentz co-variance). That is, 2 gauge potentials differing by a gauge transform are different "coordinates" for the same physical configuration. Physical Observables are "Gauge Invariant" like $F_{\mu\nu}$.

To quantize this theory we have a difficulty — since some part of A_μ doesn't enter the action, no EOM can determine it; we need some extra condition to fix it. What this condition is doesn't much matter (as long as it only fixes one field and not more) and there are many choices. As long as we only compute gauge invariant quantities we will get answers that don't depend on the choice. This is called "gauge fixing"

Some Standard Choices:

1) Coulomb Gauge $\vec{\nabla}\cdot\vec{A}=0$ (Breaks Lorentz)
2) Temporal " $A_0=0$ (" ")
3) Landau Gauge $\partial_\mu A^\mu=0$
4) Axial Gauge $A_3=0$ (Breaks Rotations)

More generally → we can also add to the Lagrangian some term that ~~B~~ violates gauge invariance and gives an EOM that fixes the gauge:

$$\delta L = -\frac{1}{2\xi}(\partial\cdot A)^2$$

3') $\xi\to 0$ requires $(\partial\cdot A)^2$: Landau
5) $\xi=1$ Feynman gauge