Entanglement properties of quantum field theory

A note of Witten's paper "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory"

Part II: The Modular Operator and Relative Entropy in Quantum Field Theory

Hao Zhang
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A Review

- The Reeh-Schlieder Theorem
- The Modular Operator and Relative Entropy

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

II. The relative modular operator

- The relative Tomita operator (Araki, 1975)

Inequalities in von Neumann algebras*

Huzihiro ARAKI

Research Institute for Mathematical Sciences
Kyoto University, Kyoto, JAPAN

Abstract Generalization of inequalities involving trace of matrices to von Neumann algebras not having traces in general is discussed.

Inequalities in Von Neumann Algebras
Araki, Huzihiro
Les rencontres physiciens-mathématiciens de Strasbourg -RCP25, Tome 22 (1975) , Exposé no. 1, 25 p.

Huzihiro Araki
荒木 不二洋
(1932/07/28-)

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

II. The relative modular operator

- **The relative Tomita operator**:

let $|\Psi\rangle$ and $|\Phi\rangle$ be both cyclic and separating (normalized) vectors for the local observable algebra $\mathfrak{A}(\mathscr{U})$ and its commutant $\mathfrak{A}(\mathscr{U})'$, the relative Tomita operator $S_{\Psi|\Phi}$ for the algebra $\mathfrak{A}(\mathscr{U})$ is defined by

$$S_{\Psi|\Phi}\left(\mathbf{a}\,|\,\Psi\rangle\right) = \mathbf{a}^{\dagger}\,|\,\Phi\rangle$$

for $\forall\ \mathbf{a} \in \mathfrak{A}(\mathscr{U})$.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

II. The relative modular operator

- **The relative Tomita operator**

1. $|\Psi\rangle$ is separating $\Rightarrow$ $S_{\Psi|\Phi}|0\rangle = 0$, the definition is consistent
2. $|\Psi\rangle$ is cyclic $\Rightarrow$ $S_{\Psi|\Phi}$ is defined on a dense subset of $\mathscr{H}$
3. The condition $|\Phi\rangle$ is also cyclic and separating makes sure the $S_{\Phi|\Psi}$ is also well-defined, and $S_{\Phi|\Psi}S_{\Psi|\Phi} = \mathbf{1}$.
4. $S'_{\Psi|\Phi} = S^{\dagger}_{\Psi|\Phi}$
5. $S_{\Psi|\Phi}|\Psi\rangle = |\Phi\rangle$

$$S_{\Psi|\Phi}\left(\mathbf{a}|\Psi\rangle\right) = \mathbf{a}^{\dagger}|\Phi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

II. The relative modular operator

- **The relative modular operator and modular conjugation**

1. The relative modular operator $\Delta_{\Psi|\Phi} = S^{\dagger}_{\Psi|\Phi} S_{\Psi|\Phi}$ is positive semidefinite, and is positive definite iff $S_{\Psi|\Phi}$ is invertible

2. If $|\Phi\rangle = |\Psi\rangle$, $\Delta_{\Psi|\Psi} = \Delta_{\Psi}$.

3. The relative modular conjugation is defined by $S_{\Psi|\Phi} = J_{\Psi|\Phi} \Delta^{1/2}_{\Psi|\Phi}$

4. If $|\Phi\rangle$ is not separating, then $S_{\Psi|\Phi}$ has a kernel, which is the same to the kernel of $\Delta_{\Psi|\Phi}$. $J_{\Psi|\Phi}$ is defined to annihilate this kernel.

5. If $|\Phi\rangle$ is not cyclic, then the image of $S_{\Psi|\Phi}$ is not dense. Then $J_{\Psi|\Phi}$ is an antiunitary map from the orthocomplement of the kernel of $S_{\Psi|\Phi}$ to its image.

$$S_{\Psi|\Phi}\left(\mathbf{a}\,|\Psi\rangle\right) = \mathbf{a}^{\dagger}\,|\Phi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

II. The relative modular operator

- Behavior under unitary transformation:

$$S_{\Psi|\Phi}\left(\mathbf{a}\,|\,\Psi\rangle\right) = \mathbf{a}^\dagger\,|\,\Phi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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- Behavior under unitary transformation:

$$|\,\Phi\rangle \to \mathbf{a}'|\,\Phi\rangle, \;\; \mathbf{a}'^{\dagger}\mathbf{a}' = \mathbf{a}'\mathbf{a}'^{\dagger} = \mathbf{1}' \text{ (unit in } \mathfrak{A}(\mathscr{U})'\text{)}$$

$$S_{\Psi|\Phi}\,(\mathbf{a}\,|\,\Psi\rangle) = \mathbf{a}^{\dagger}\,|\,\Phi\rangle$$

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$$\Rightarrow \;\; S_{\Psi|\mathbf{a}'\Phi}\mathbf{a}|\Psi\rangle = \mathbf{a}^{\dagger}(\mathbf{a}'|\Phi\rangle) = \mathbf{a}'\mathbf{a}^{\dagger}|\Phi\rangle = \mathbf{a}'S_{\Psi|\Phi}\mathbf{a}|\Psi\rangle$$

$$S_{\Psi|\Phi}\left(\mathbf{a}|\Psi\rangle\right) = \mathbf{a}^{\dagger}|\Phi\rangle$$

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$$\Rightarrow \;\; S_{\Psi|\mathbf{a}'\Phi} = \mathbf{a}'S_{\Psi|\Phi}$$

$$S_{\Psi|\Phi}\,(\mathbf{a}\,|\,\Psi\rangle) = \mathbf{a}^{\dagger}\,|\,\Phi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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$$\Rightarrow \quad S_{\Psi|\mathbf{a}'\Phi} = \mathbf{a}'S_{\Psi|\Phi}$$

$$\therefore \quad \Delta_{\Psi|\mathbf{a}'\Phi} = S^{\dagger}_{\Psi|\mathbf{a}'\Phi}S_{\Psi|\mathbf{a}'\Phi} = S^{\dagger}_{\Psi|\Phi}\mathbf{a}'^{\dagger}\mathbf{a}'S_{\Psi|\Phi} = S^{\dagger}_{\Psi|\Phi}S_{\Psi|\Phi} = \Delta_{\Psi|\Phi}$$

$$S_{\Psi|\Phi}\,(\mathbf{a}|\Psi\rangle) = \mathbf{a}^{\dagger}|\Phi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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$$\therefore \;\; \Delta_{\Psi|\mathbf{a}'\Phi} = S^{\dagger}_{\Psi|\mathbf{a}'\Phi}S_{\Psi|\mathbf{a}'\Phi} = S^{\dagger}_{\Psi|\Phi}\mathbf{a}'^{\dagger}\mathbf{a}'S_{\Psi|\Phi} = S^{\dagger}_{\Psi|\Phi}S_{\Psi|\Phi} = \Delta_{\Psi|\Phi}$$

$$\boxed{S_{\Psi|\mathbf{a}'\Phi} = \mathbf{a}'S_{\Psi|\Phi}, \;\; \Delta_{\Psi|\mathbf{a}'\Phi} = \Delta_{\Psi|\Phi}}$$

$$\boxed{S_{\Psi|\Phi}\,(\mathbf{a}|\Psi\rangle) = \mathbf{a}^{\dagger}|\Phi\rangle}$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

II. The relative modular operator

- The relative modular operator transformation:

$$S_{\Psi|\Phi}\left(\mathbf{a}\,|\,\Psi\rangle\right) = \mathbf{a}^{\dagger}\,|\,\Phi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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$$\langle \mathbf{a}^\dagger \Psi \,|\, \Delta_{\Psi|\Phi} \,|\, \mathbf{b}\Psi \rangle = \langle \mathbf{a}^\dagger \Psi \,|\, S^\dagger_{\Psi|\Phi} S_{\Psi|\Phi} \,|\, \mathbf{b}\Psi \rangle = \langle S_{\Psi|\Phi} \mathbf{b}\Psi \,|\, S_{\Psi|\Phi} \mathbf{a}^\dagger \Psi \rangle$$

$$S_{\Psi|\Phi} \left(\mathbf{a} \,|\, \Psi \rangle \right) = \mathbf{a}^\dagger \,|\, \Phi \rangle$$

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$$= \langle \mathbf{b}^\dagger \Phi \,|\, \mathbf{a}\Phi \rangle$$

$$S_{\Psi|\Phi} \left(\mathbf{a} \,|\, \Psi \rangle \right) = \mathbf{a}^\dagger \,|\, \Phi \rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

- Relative entropy in classical information theory: Kullback-Leibler divergence (1951)

Solomon Kullback
(1907/04/03-1994/08/05)

Richard A. Leibler
(1914/03/18-2003/10/25)

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

- Relative entropy in classical information theory: [Kullback-Leibler divergence (1951)]

Solomon Kullback
(1907/04/03-1994/08/05)

$$D_{KL}(P\|Q) = \int_E p(x)\big(\log p(x) - \log q(x)\big)\, dx$$

Richard A. Leibler
(1914/03/18-2003/10/25)

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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- "A degree of surprising" $\langle \log(1/p) \rangle$: an example, distributions on n-state.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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$$p_1(i) = 1/n$$

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$$p_1(i) = 1/n \qquad S_1 = -\sum_{i=1}^{n} p_1(i) \log p_1(i) = \log n$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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$$S(1\|2) = -\sum_{i=1}^{n} p_1(i) \log p_2(i) - \left[-\sum_{i=1}^{n} p_1(i) \log p_1(i) \right] \to \infty$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

- Relative entropy in classical information theory: [Kullback-Leibler divergence (1951)]
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$$S(2\|1) = -\sum_{i=1}^{n} p_2(i) \log p_1(i) - \left[-\sum_{i=1}^{n} p_2(i) \log p_2(i) \right] = \log n$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

- Relative entropy in quantum mechanics: [Hisaharu Umegaki (梅垣寿春，1962)]

CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA, IV
(ENTROPY AND INFORMATION)

BY HISAHARU UMEGAKI

1. Introduction.

The theory of information, created by Shannon [23], is developed by Feinstein, Kullback, MacMillan, Wiener and other American statisticians (e. g., cf. [10]), and also advanced into the ergodic theory by Gelfand, Khinchin, Kolmogorov, Yaglom and other Russian probabilists (e. g., cf. [8]). Through recent years, the theory is regarded as a new chapter in the theory of probability.

Recently, Segal [22] gave a mathematical formulation of the entropy of state of a von Neumann algebra, which contains both the cases for the theory of information and the theory of quantum statistics. Segal's theorem was reformulated in operator algebraic form by Nakamura and Umegaki [16] and independently by Davis [3].

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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- Relative entropy in quantum mechanics: Hisaharu Umegaki (梅垣寿春，1962)

CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA, IV
(ENTROPY AND INFORMATION)

By Hisaharu Umegaki

1.

$$I(\rho\|\sigma) = \mathbf{Tr}[\rho(\log \rho - \log \sigma)]$$

T Fein-
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THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

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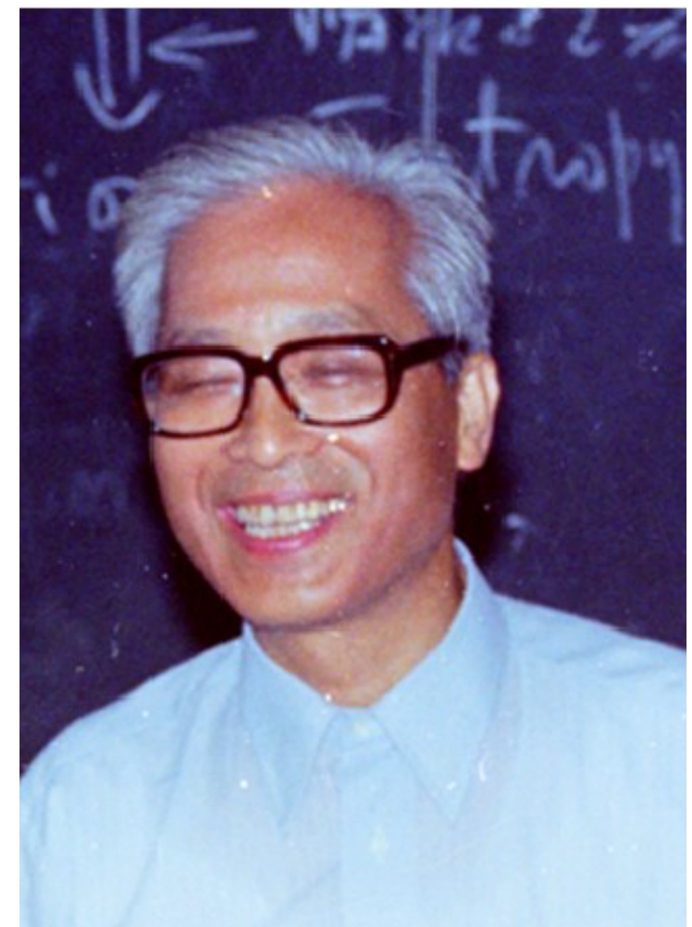

Hisaharo Umegaki
梅垣 寿春
（うめがき ひさはる）
(1925-2012/05/22)

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

- Relative entropy in quantum field theory: Huzihiro Araki (荒木不二洋，1975-1976)

Huzihiro Araki
荒木 不二洋
(1932/07/28-)

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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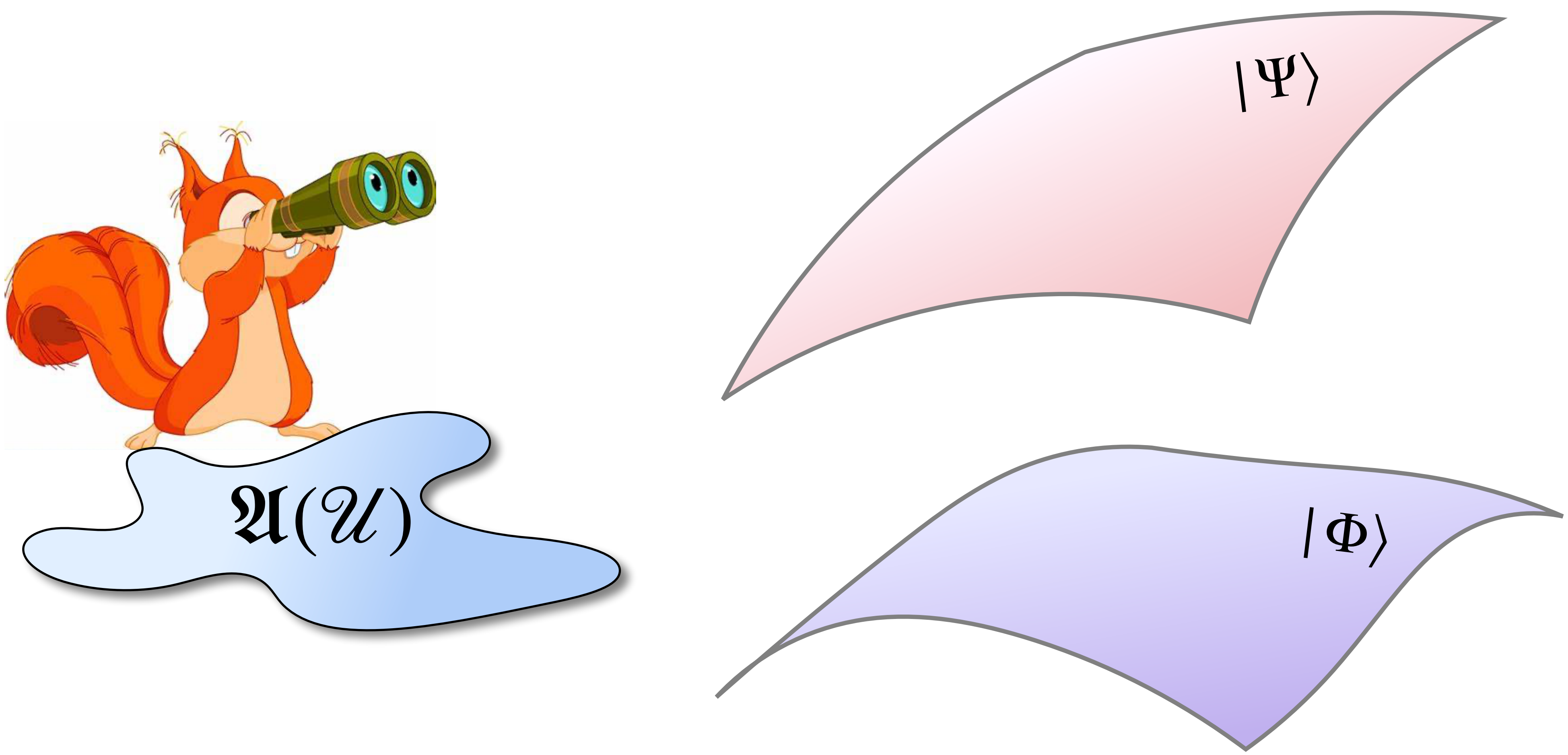

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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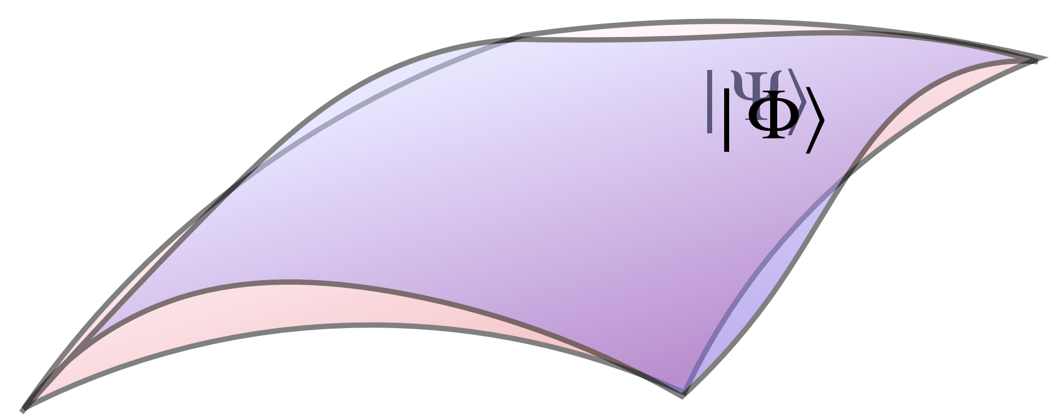

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

- Relative entropy in quantum field theory:

The relative entropy $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ between two states $|\Psi\rangle$ and $|\Phi\rangle$, for measurements in the region $\mathcal{U}$, is

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

- Basic properties of the relative entropy

1. $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ is a real number or $+\infty$;

2. If $|\Phi\rangle$ is not a separating vector of $\mathfrak{A}(\mathcal{U})$, 0 is a eigenvalue of $\Delta_{\Psi|\Phi;\mathcal{U}}$, then $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ may be $+\infty$;

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle$$

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2. If $|\Phi\rangle$ is not a separating vector of $\mathfrak{A}(\mathscr{U})$, 0 is a eigenvalue of $\Delta_{\Psi|\Phi;\mathscr{U}}$, then $\mathscr{S}_{\Psi|\Phi}(\mathscr{U})$ may be $+\infty$;

3. **$\mathscr{S}_{\Psi|\Phi}(\mathscr{U})$ is always non-negative.**

$$\mathscr{S}_{\Psi|\Phi}(\mathscr{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathscr{U}}|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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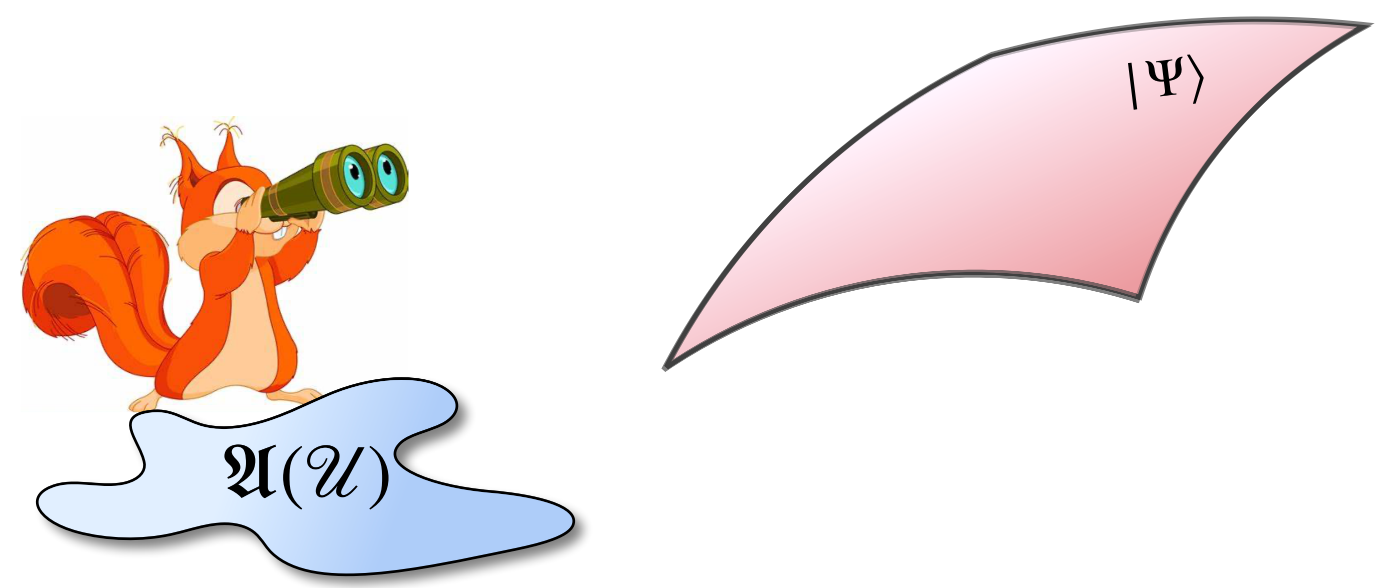

$$\mathscr{S}_{\Psi|\Phi}(\mathscr{U}) = -\langle\Psi\,|\log\Delta_{\Psi|\Phi;\mathscr{U}}\,|\Psi\rangle$$

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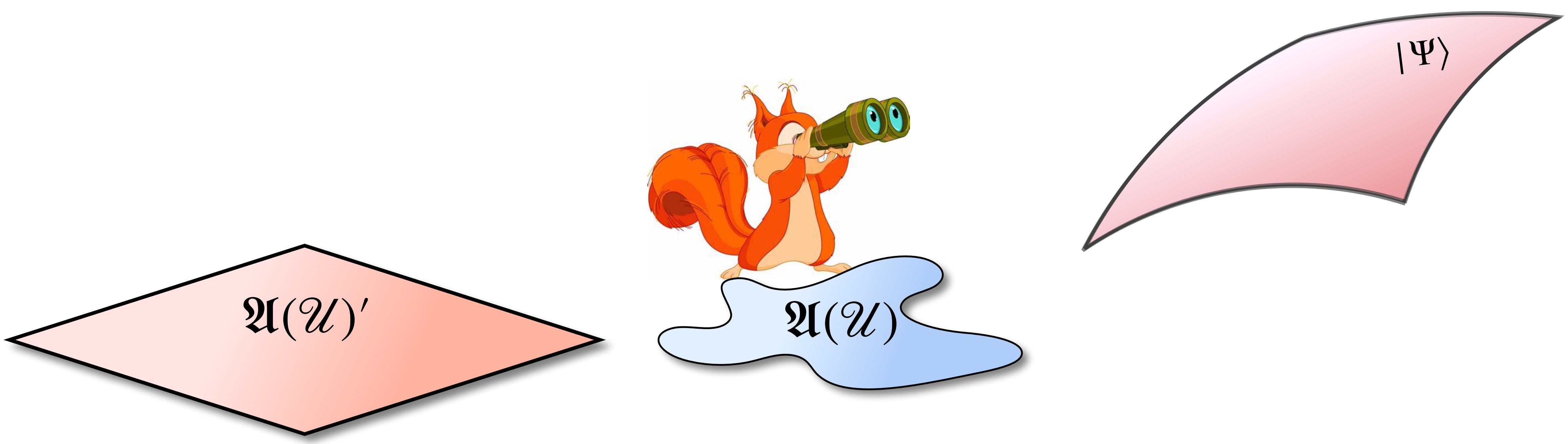

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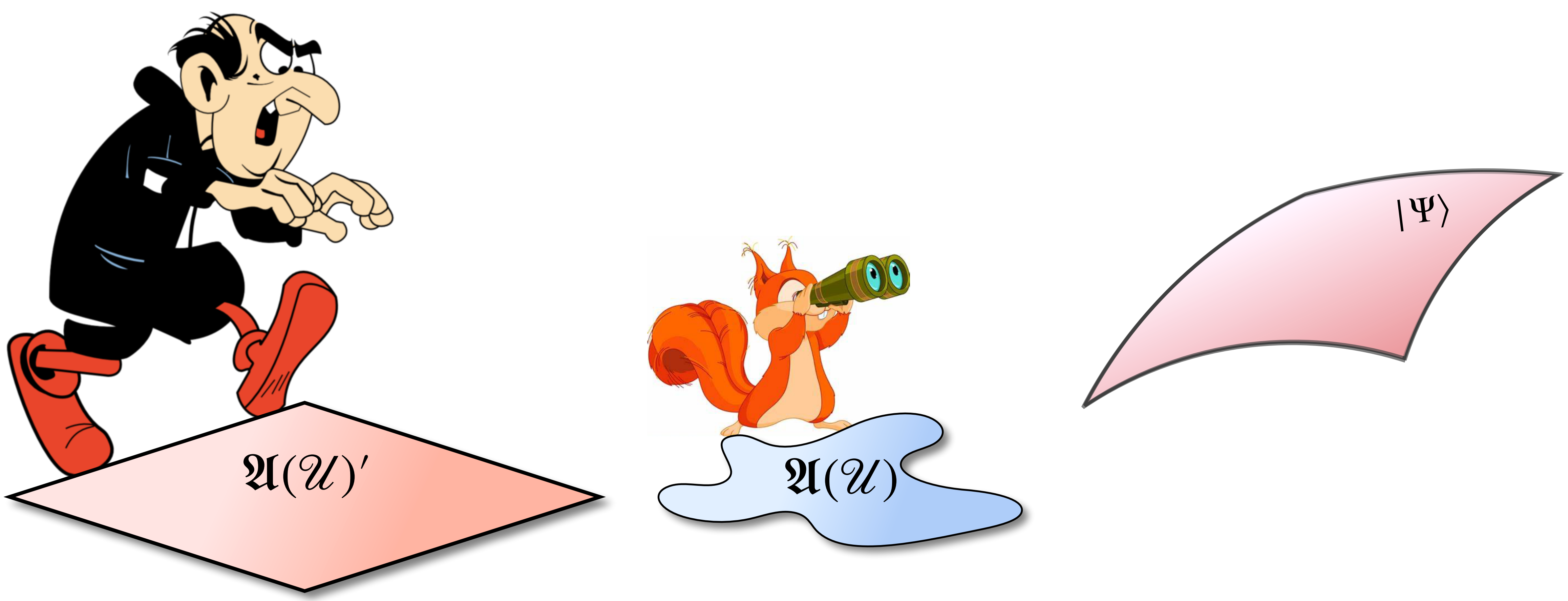

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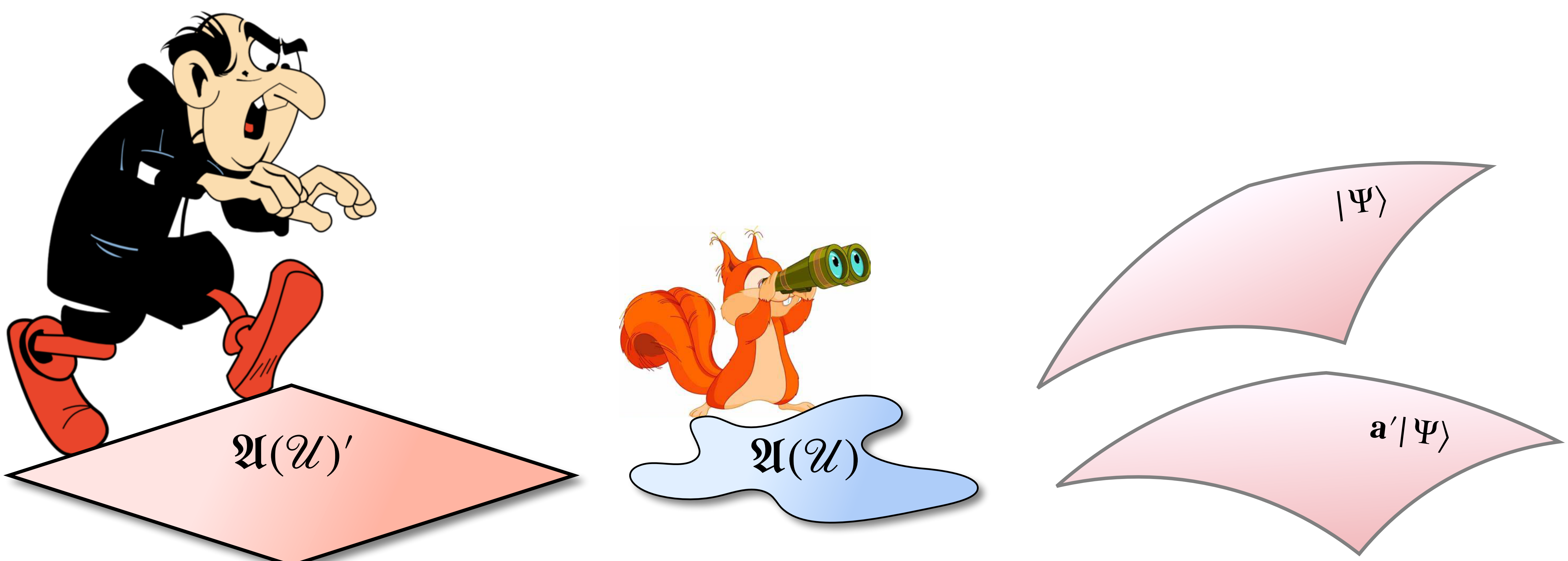

$$\mathscr{S}_{\Psi|\Phi}(\mathscr{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathscr{U}}|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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If $|\Phi\rangle = \mathbf{a}'|\Psi\rangle$, $\mathbf{a}' \in \mathfrak{A}(\mathscr{U})'$, $\mathbf{a}'^{\dagger}\mathbf{a}' = \mathbf{1}'$, then $\Delta_{\Psi|\Phi;\mathscr{U}} = \Delta_{\Psi|\Psi} = \Delta_{\Psi}$, so we have $f(\Delta_{\Psi})|\Psi\rangle = f(1)|\Psi\rangle$. That proves

$$\mathscr{S}_{\Psi|\Phi}(\mathscr{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathscr{U}}|\Psi\rangle$$

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$$\mathscr{S}_{\Psi|\mathbf{a}'\Psi}(\mathscr{U}) = -\langle\Psi|\log(1)|\Psi\rangle = 0$$

$$\mathscr{S}_{\Psi|\Phi}(\mathscr{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathscr{U}}|\Psi\rangle$$

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$$\mathcal{S}_{\Psi|\mathbf{a}'\Psi}(\mathcal{U}) = -\langle\Psi|\log(1)|\Psi\rangle = 0$$

$$\mathcal{S}_{\Psi|\Psi}(\mathcal{U}) = -\langle\Psi|\log(1)|\Psi\rangle = 0$$

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

III. Relative entropy in quantum field theory

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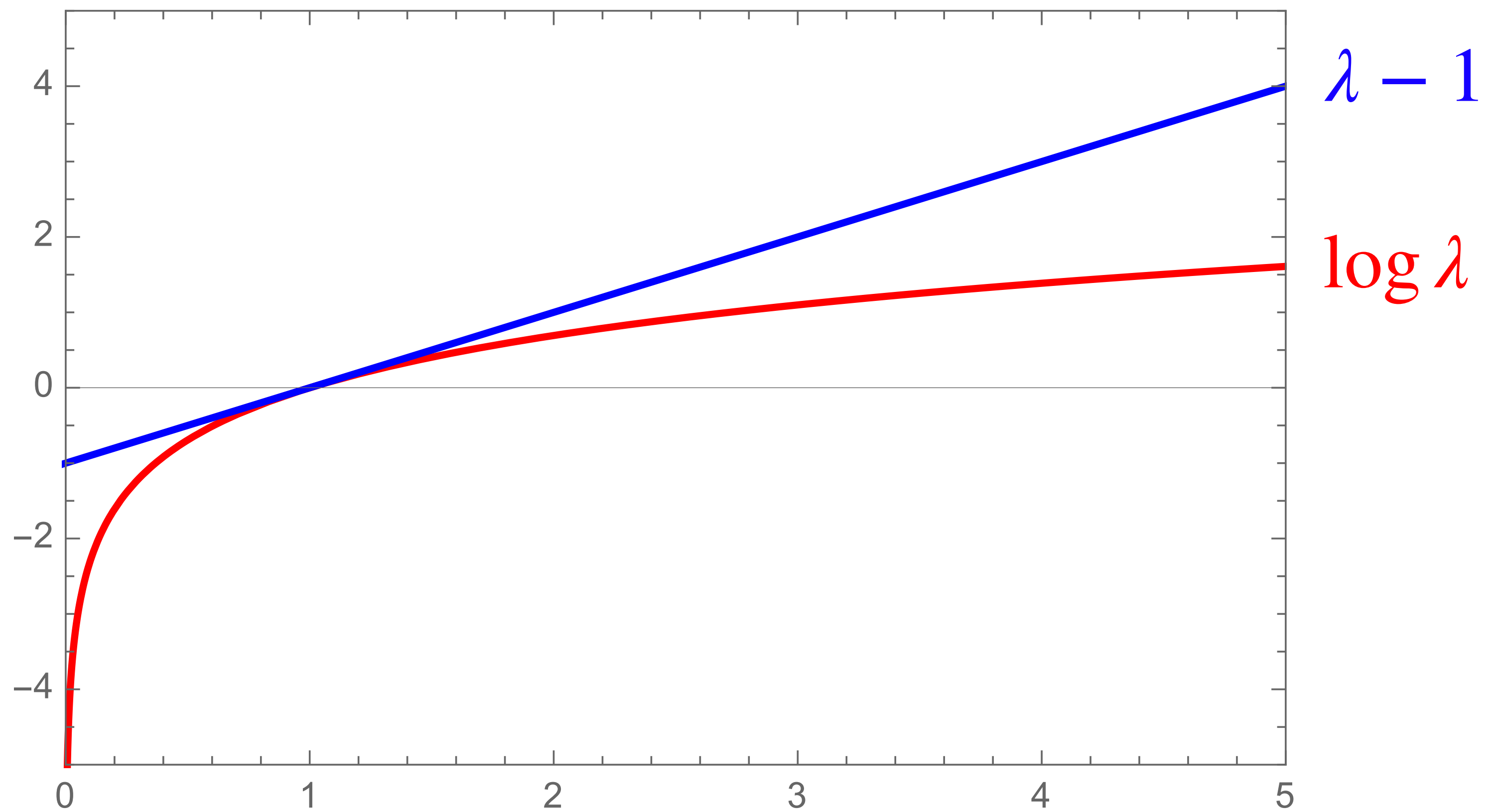
3. $\mathcal{S}_{\Psi|\Phi}(\mathcal{U})$ **is always non-negative.**

To show $\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) > 0$ for $|\Phi\rangle \neq \mathbf{a}'|\Psi\rangle$, one uses $\log\lambda \leqslant \lambda - 1$ $(\lambda > 0)$



$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle$$

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$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle \geqslant \langle\Psi|(\mathbf{1}-\Delta_{\Psi|\Phi;\mathcal{U}})|\Psi\rangle$$

$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle$

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3. $\mathscr{S}_{\Psi|\Phi}(\mathscr{U})$ **is always non-negative.**

$$\mathscr{S}_{\Psi|\Phi}(\mathscr{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathscr{U}}|\Psi\rangle \geqslant \langle\Psi|(\mathbf{1}-\Delta_{\Psi|\Phi;\mathscr{U}})|\Psi\rangle$$

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$$= \langle\Psi|\Psi\rangle - \langle\Phi|\Phi\rangle = 0$$

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$\mathbf{a}$ is linear $\;\Rightarrow\;$ $\mathbf{a}'$ is linear

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$$\forall\; \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{A}(\mathscr{U}), \;\; \langle \mathbf{c}\Psi \,|\, \mathbf{a}\mathbf{a}' \,|\, \mathbf{b}\Psi \rangle = \langle \mathbf{c}\Psi \,|\, \mathbf{a}\mathbf{a}'\mathbf{b} \,|\, \Psi \rangle = \langle \mathbf{a}^\dagger \mathbf{c}\Psi \,|\, \mathbf{b}\Phi \rangle$$

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$$\begin{aligned} \forall\; \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{A}(\mathcal{U}), \;\; \langle \mathbf{c}\Psi \,|\, \mathbf{a}\mathbf{a}' \,|\, \mathbf{b}\Psi \rangle &= \langle \mathbf{c}\Psi \,|\, \mathbf{a}\mathbf{a}'\mathbf{b} \,|\, \Psi \rangle = \langle \mathbf{a}^\dagger \mathbf{c}\Psi \,|\, \mathbf{b}\Phi \rangle \\ &= \langle \mathbf{c}\Psi \,|\, \mathbf{a}\mathbf{b}\Phi \rangle = \langle \mathbf{c}\Psi \,|\, \mathbf{a}'\mathbf{a} \,|\, \mathbf{b}\Psi \rangle \;\;\Rightarrow\;\; \mathbf{a}' \in \mathfrak{A}(\mathcal{U})' \end{aligned}$$

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  3. $\mathscr{S}_{\Psi|\Phi}(\mathscr{U})$ **is always non-negative. It is zero iff there is an unitary operator** $\mathbf{a}' \in \mathfrak{A}(\mathscr{U})'$ **satisfies** $|\,\Phi\rangle = \mathbf{a}'|\,\Psi\rangle$

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- The positivity of relative entropy is a very important.

- Another key property of relative entropy is monotonicity.

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle$$

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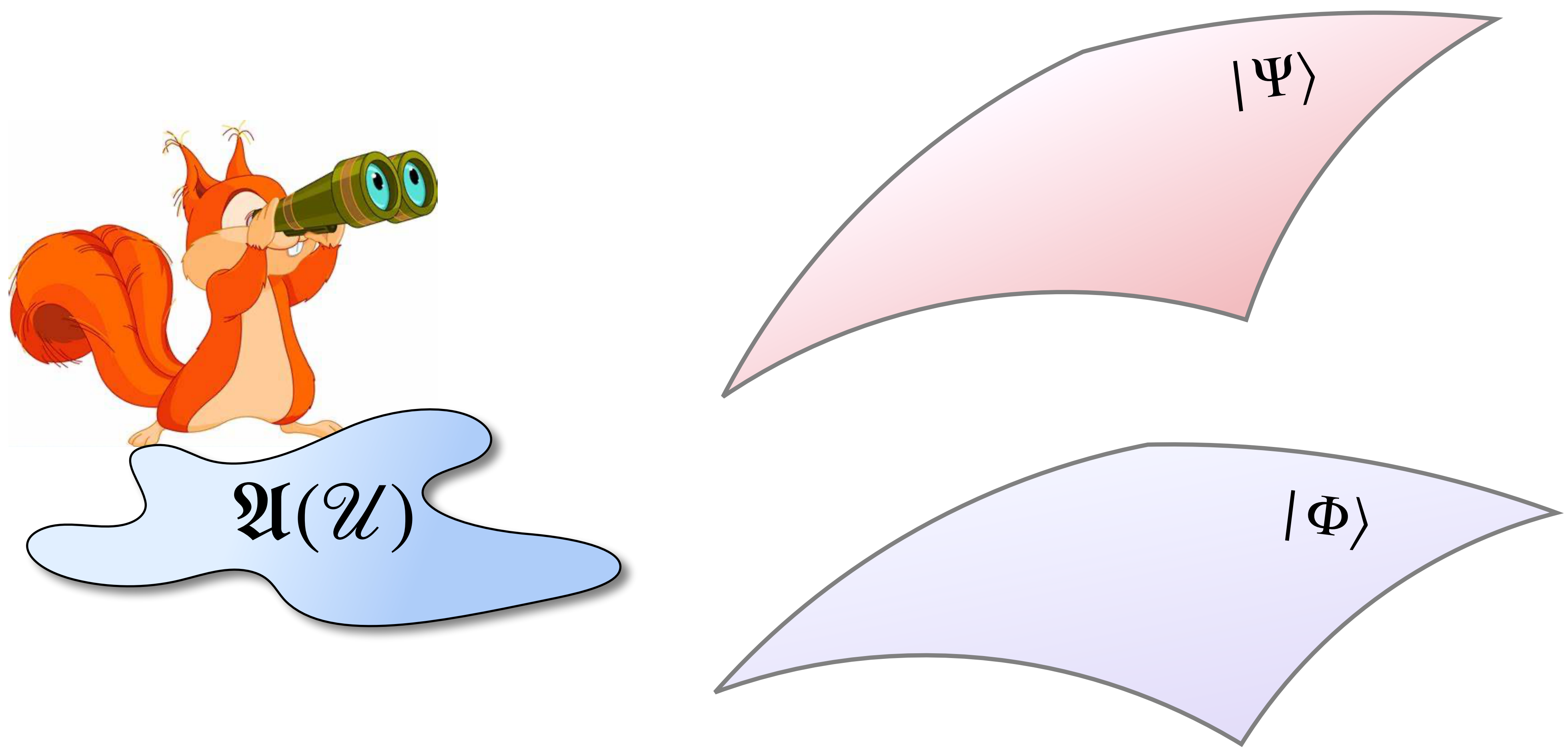
IV. Monotonicity of relative entropy

- Monotonicity of relative entropy



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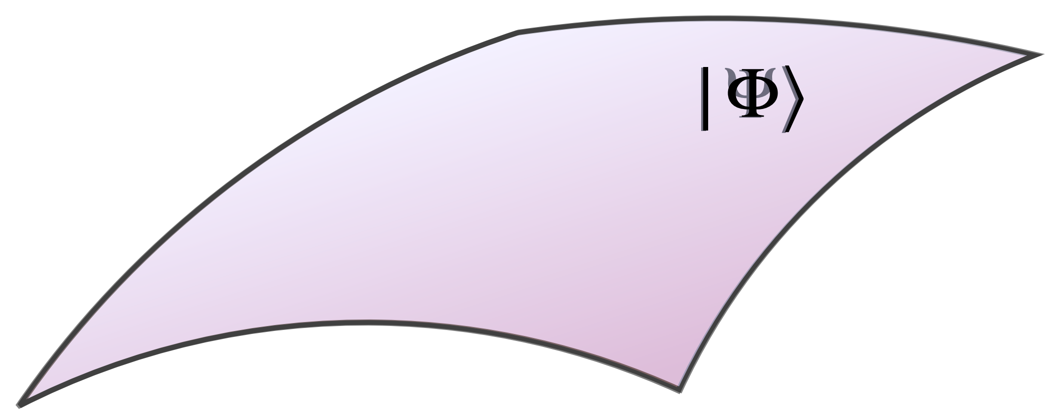

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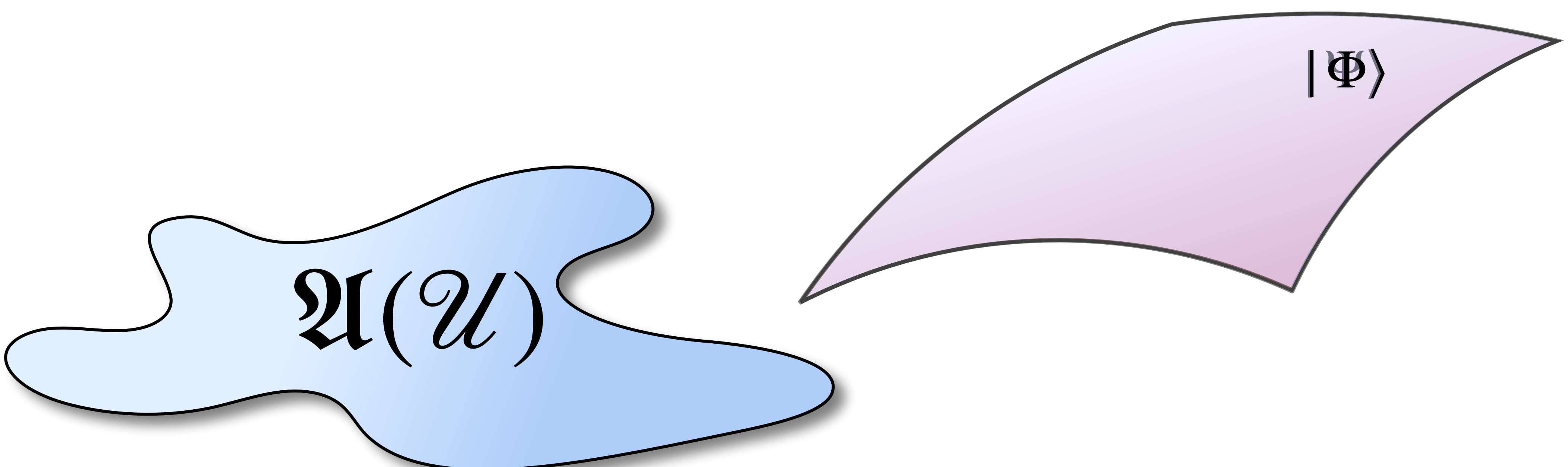

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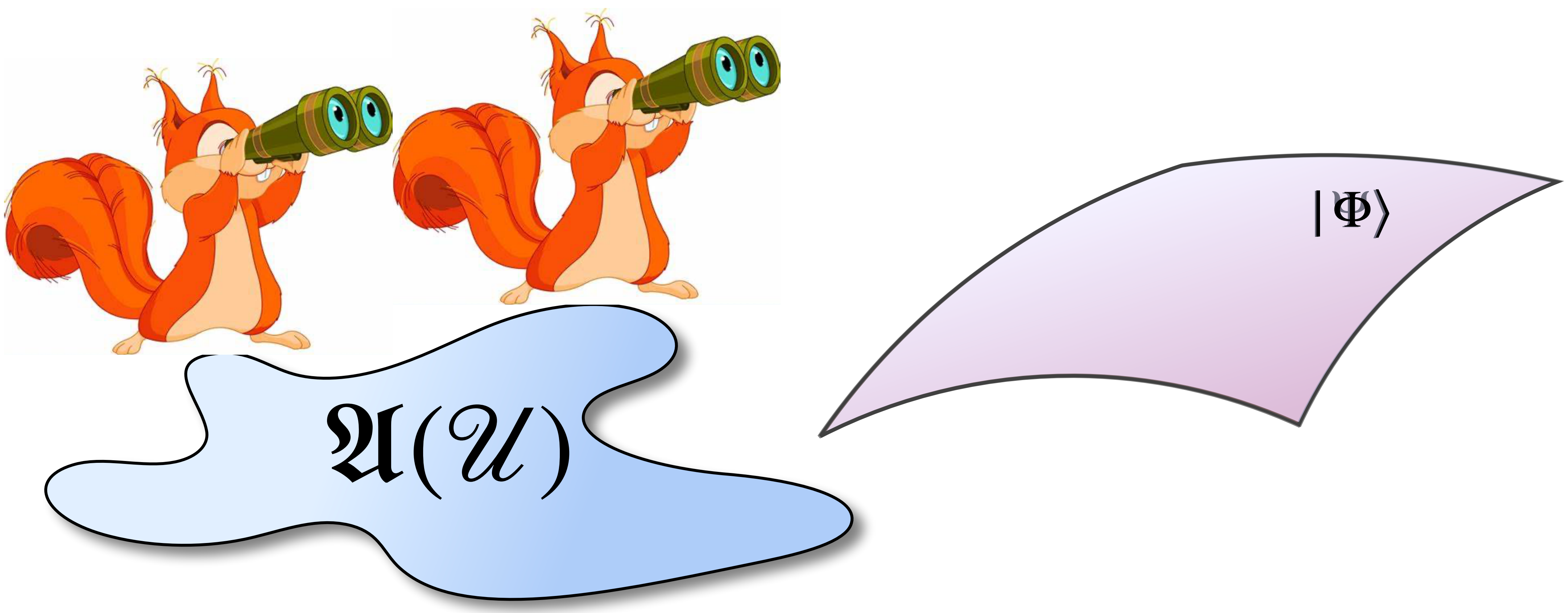

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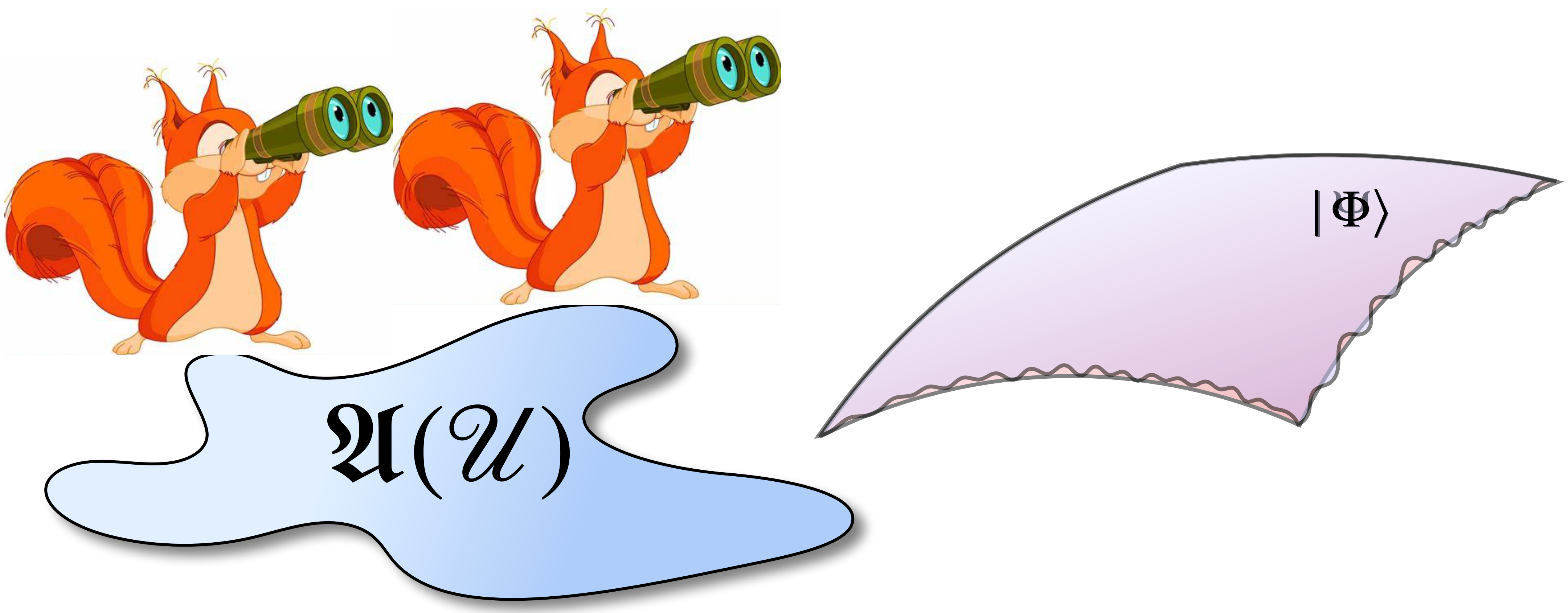

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IV. Monotonicity of relative entropy

- Monotonicity of relative entropy:

  If $\tilde{\mathscr{U}} \subset \mathscr{U}$, then

$$\mathscr{S}_{\Psi|\Phi}(\tilde{\mathscr{U}}) = -\langle\Psi\,|\log\Delta_{\Psi|\Phi;\tilde{\mathscr{U}}}\,|\Psi\rangle \leqslant \mathscr{S}_{\Psi|\Phi}(\mathscr{U}) = -\langle\Psi\,|\log\Delta_{\Psi|\Phi;\mathscr{U}}\,|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

IV. Monotonicity of relative entropy

- Monotonicity of relative entropy
  - The monotonicity is a direct consequence of the relation

$$\boxed{\tilde{\mathscr{U}} \subset \mathscr{U} \;\;\Rightarrow\;\; \Delta_{\Psi|\Phi;\tilde{\mathscr{U}}} \geqslant \Delta_{\Psi|\Phi;\mathscr{U}}}$$

  - What does it mean? And why is it sufficient?

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  - **Positive operator**: a self-adjoint operator P is called positive iff $\langle\psi|P|\psi\rangle \geqslant 0$ for any $|\psi\rangle \in \mathscr{H}$;

  - If P and Q are both **bounded** self-adjoint operators, $P \geqslant Q$ means $P - Q \geqslant 0$;

  - How about generic P and Q?

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IV. Monotonicity of relative entropy

- Monotonicity of relative entropy
  - If $P, Q \geqslant 0$,
$$P \geqslant Q \;\; \Leftrightarrow \;\; \forall \; s > 0, \; \frac{1}{s+P} \leqslant \frac{1}{s+Q}$$
  - Proof: "⇒", consider a (one-real-parameter) family of operators $R(t) = tP + (1-t)Q$, $t \in \mathbb{R}$, then $\dot{R} = dR/dt = P - Q \geqslant 0$, and

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$$\frac{d}{dt}\frac{1}{s+R(t)} = -\frac{1}{s+R(t)}\dot{R}\frac{1}{s+R(t)} \leqslant 0$$

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$$\frac{d}{dt}\frac{1}{s+R(t)} = -\frac{1}{s+R(t)}\dot{R}\frac{1}{s+R(t)} \leqslant 0$$

$$\Rightarrow\;\; \frac{1}{s+P} = \frac{1}{s+R(1)} \leqslant \frac{1}{s+R(0)} = \frac{1}{s+Q}$$

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  - If $P, Q \geqslant 0$,
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  - Proof: "$\Leftarrow$" (same method)

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IV. Monotonicity of relative entropy

- Monotonicity of relative entropy
  - For general (unbounded) non-negative operator , it is reasonable to define If $P \geqslant Q$ by

$$\langle\psi|\frac{1}{s+P}|\psi\rangle \leqslant \langle\psi|\frac{1}{s+Q}|\psi\rangle, \ \ \forall\ |\psi\rangle \in \mathscr{H},\ s \in \mathbb{R}^{+}$$

  - Because $1/(s+P)$ and $1/(s+Q)$ are bounded and hence could be defined in the whole Hilbert space, this is a much stronger and more useful statement than just saying that $\langle\psi|P|\psi\rangle \geqslant \langle\psi|Q|\psi\rangle$ for all $|\psi\rangle$ on which both P and Q are defined.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

IV. Monotonicity of relative entropy

- Monotonicity of relative entropy
  - Because $1/(s+R)$ is a decreasing function of (positive) R,

$$\log R = \int_0^{+\infty} ds \left(\frac{1}{s+1} - \frac{1}{s+R} \right)$$

  is an increasing function of R.

  - This proves that $P \geqslant Q$ (or $1/(s+P) \leqslant 1/(s+Q)$) implies

$$\log P \geqslant \log Q$$

  - So $\Delta_{\Psi|\Phi;\tilde{\mathscr{U}}} \geqslant \Delta_{\Psi|\Phi;\mathscr{U}} \ \Rightarrow \ \log \Delta_{\Psi|\Phi;\tilde{\mathscr{U}}} \geqslant \log \Delta_{\Psi|\Phi;\mathscr{U}}$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

IV. Monotonicity of relative entropy

- Monotonicity of relative entropy
  - Another useful inequality

$$R^\alpha = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} s^\alpha \left(\frac{1}{s} - \frac{1}{s+R} \right) ds, \quad 0 < \alpha < 1$$

$$\frac{d}{dt} R^\alpha = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} s^\alpha \frac{1}{s+R} \dot{R} \frac{1}{s+R} ds$$

$$\therefore \ \dot{R} \geqslant 0 \ \Rightarrow \ R^\alpha \nearrow$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

IV. Monotonicity of relative entropy

- Monotonicity of relative entropy

  - Another useful inequality

$$R^\alpha = R \cdot R^\beta = \frac{\sin \pi(\alpha - 1)}{\pi} \int_0^{+\infty} s^{\alpha-1} \left(\frac{R}{s} - 1 + \frac{s}{s+R} \right) ds, \quad 1 < \alpha < 2$$

$$\frac{d}{dt} R^\alpha = \frac{\sin \pi(\alpha - 1)}{\pi} \int_0^{+\infty} s^{\alpha-1} \left(\frac{\dot{R}}{s} - s \frac{1}{s+R} \dot{R} \frac{1}{s+R} \right) ds$$

$$\therefore \;\; \dot{R} \geqslant 0 \;\; \not\Rightarrow \;\; R^\alpha \nearrow$$

  - For example:

$$R = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; \dot{R} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \; |\chi\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \Rightarrow \; \langle \chi | \frac{d}{dt} R^\alpha | \chi \rangle < 0$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- Why monotonicity?
- If $|\Psi\rangle$ is cyclic vector for both $\mathfrak{A}(\mathscr{U})$ and $\mathfrak{A}(\tilde{\mathscr{U}})$:

$$\Delta_{\Psi|\Phi;\mathscr{U}} = S^{\dagger}_{\Psi|\Phi;\mathscr{U}} S_{\Psi|\Phi;\mathscr{U}}, \quad S_{\Psi|\Phi;\mathscr{U}} : \ \mathbf{a}\,|\Psi\rangle \mapsto \mathbf{a}^{\dagger}\,|\Phi\rangle, \quad \forall\ \mathbf{a} \in \mathfrak{A}(\mathscr{U})$$
$$\Delta_{\Psi|\Phi;\tilde{\mathscr{U}}} = S^{\dagger}_{\Psi|\Phi;\tilde{\mathscr{U}}} S_{\Psi|\Phi;\tilde{\mathscr{U}}}, \quad S_{\Psi|\Phi;\tilde{\mathscr{U}}} : \ \mathbf{a}\,|\Psi\rangle \mapsto \mathbf{a}^{\dagger}\,|\Phi\rangle, \quad \forall\ \mathbf{a} \in \mathfrak{A}(\tilde{\mathscr{U}})$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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$$\Delta_{\Psi|\Phi;\tilde{\mathscr{U}}} = S^{\dagger}_{\Psi|\Phi;\tilde{\mathscr{U}}} S_{\Psi|\Phi;\tilde{\mathscr{U}}}, \quad S_{\Psi|\Phi;\tilde{\mathscr{U}}} : \ \mathbf{a}\,|\Psi\rangle \mapsto \mathbf{a}^{\dagger}\,|\Phi\rangle, \quad \forall\ \mathbf{a} \in \mathfrak{A}(\tilde{\mathscr{U}})$$

What is the difference?

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

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- If $|\Psi\rangle$ is cyclic vector for both $\mathfrak{A}(\mathscr{U})$ and $\mathfrak{A}(\tilde{\mathscr{U}})$:

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$$\Delta_{\Psi|\Phi;\tilde{\mathscr{U}}} = S^{\dagger}_{\Psi|\Phi;\tilde{\mathscr{U}}} S_{\Psi|\Phi;\tilde{\mathscr{U}}}, \quad S_{\Psi|\Phi;\tilde{\mathscr{U}}} : \ \mathbf{a}\,|\Psi\rangle \mapsto \mathbf{a}^{\dagger}\,|\Phi\rangle, \quad \forall\ \mathbf{a} \in \mathfrak{A}(\tilde{\mathscr{U}})$$

Domains!

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- Why monotonicity?
- If $|\Psi\rangle$ is cyclic vector for both $\mathfrak{A}(\mathscr{U})$ and $\mathfrak{A}(\tilde{\mathscr{U}})$:

$$\Delta_{\Psi|\Phi;\mathscr{U}} = S^{\dagger}_{\Psi|\Phi;\mathscr{U}} S_{\Psi|\Phi;\mathscr{U}}, \quad S_{\Psi|\Phi;\mathscr{U}} : \ \mathbf{a}\,|\,\Psi\rangle \mapsto \mathbf{a}^{\dagger}\,|\,\Phi\rangle, \quad \forall\ \mathbf{a} \in \mathfrak{A}(\mathscr{U})$$
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Domains!

And so what??

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- $\tilde{\mathscr{U}} \subset \mathscr{U} \;\Rightarrow\; \mathfrak{A}(\tilde{\mathscr{U}}) \subset \mathfrak{A}(\mathscr{U})$

- The domain of $S_{\Psi|\Phi;\mathscr{U}}$ is larger than the domain of $S_{\Psi|\Phi;\tilde{\mathscr{U}}}$.

- Extension of operator:

Let X, Y be unbounded operators on a Hilbert space $\mathscr{H}$ (either both linear or both antilinear). If $\mathbf{Dom}(Y) \subset \mathbf{Dom}(X)$ and $Y|_{\mathbf{Dom}(Y)} = X|_{\mathbf{Dom}(Y)}$, then X is called an extension of Y (and usually written as $Y \subset X$).

$$Y \subset X \;\Rightarrow\; X^\dagger X \leqslant Y^\dagger Y \;\Rightarrow\; \log X^\dagger X \leqslant \log Y^\dagger Y$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- The definition of $X^\dagger X$:

Define the (positive definite) Hermitian form $F(|\chi\rangle, |\eta\rangle) = \langle X\chi | X\eta\rangle$, $\forall\ |\chi\rangle, |\eta\rangle \in \mathbf{Dom}(X)$, if $\langle\zeta|\psi\rangle = \langle X\xi | X\psi\rangle$ ($\langle\zeta|\psi\rangle = \langle X\psi | X\xi\rangle$ for antilinear X) holds for $\forall\ |\psi\rangle \in \mathbf{Dom}(X)$, one defines

$$X^\dagger X|\xi\rangle = |\zeta\rangle$$

- If two Hermitian form F and G on $\mathscr{H}$ agree where they are both defined and F is defined whenever G is defined. Then F is called an extension of G.

- In our problem, $S_{\Psi|\Phi;\mathscr{U}}$ is an extension of $S_{\Psi|\Phi;\tilde{\mathscr{U}}}$, $\Delta_{\Psi|\Phi;\mathscr{U}} = S^\dagger_{\Psi|\Phi;\mathscr{U}} S_{\Psi|\Phi;\mathscr{U}}$ is an extension of $\Delta_{\Psi|\Phi;\tilde{\mathscr{U}}} = S^\dagger_{\Psi|\Phi;\tilde{\mathscr{U}}} S_{\Psi|\Phi;\tilde{\mathscr{U}}}$.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- To understand $Y \subset X \Rightarrow X^\dagger X \leqslant Y^\dagger Y$, Witten gives an example of n-dim quantum mechanics. We would like to simplify it to a 1-dim quantum mechanics example.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- Consider a free particle in 1-dim space region $[0,\ 1]$, the Hilbert space is $L^2_{\mathbb{C}}(0,\ 1)$.

- The momentum operator could be defined either only for the wave functions with Dirichlet boundary condition $P_0 = -id_x$, or for general wave functions $P_1 = -id_x$.

- It is obviously that P_1 is an extension of P_0.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- The Hermitian forms are

$$F_0(\psi_1, \psi_2) = \langle P_0\psi_1 \,|\, P_0\psi_2 \rangle = \int_0^1 \frac{d\bar{\psi}_1}{dx}\frac{d\psi_2}{dx}dx$$

$$F_1(\psi_1, \psi_2) = \langle P_1\psi_1 \,|\, P_1\psi_2 \rangle = \int_0^1 \frac{d\bar{\psi}_1}{dx}\frac{d\psi_2}{dx}dx$$

- Is $P_i^\dagger P_i = -d_x^2 = \Delta$ the Laplacian operator?

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- The Hermitian forms are

$$F_i(\psi_1, \psi_2) = \int_0^1 \frac{d\bar{\psi}_1}{dx}\frac{d\psi_2}{dx}dx = \frac{d\bar{\psi}_1}{dx}\psi_2\Big|_{x=1} - \frac{d\bar{\psi}_1}{dx}\psi_2\Big|_{x=0} + \int_0^1 \left(-\frac{d^2\bar{\psi}_1}{dx^2}\right)\psi_2 dx$$

- For F_0, because the wave functions satisfy the Dirichlet boundary condition, $\Delta_D = P_0^\dagger P_0$ is called the Dirichlet Laplacian

$$\langle P_0^\dagger P_0 \psi_1 | \psi_2 \rangle = \langle \Delta_D \psi_1 | \psi_2 \rangle = \int_0^1 \left(-\frac{d^2\bar{\psi}_1}{dx^2}\right)\psi_2 dx$$

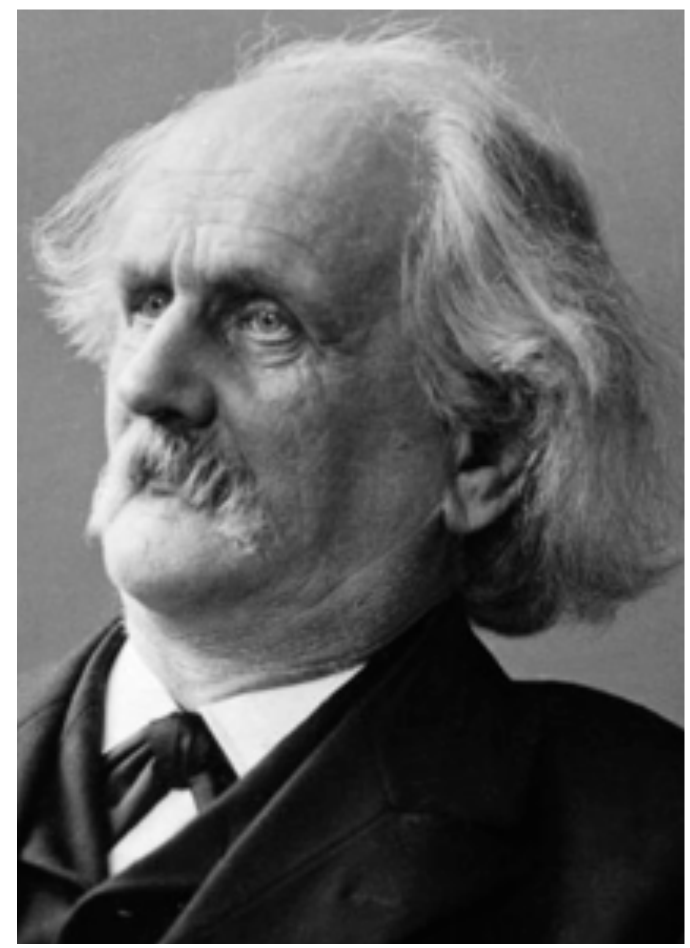
- For F_1, because the wave functions do not satisfy the Dirichlet boundary condition, to ensure the contribution from first two terms vanishes, $\Delta_N = P_1^\dagger P_1$ can be defined only on the wave functions with $d\psi_1(0)/dx = d\psi_1(1)/dx = 0$, which is so called the Neumann Laplacian.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- Does $\Delta_D \geqslant \Delta_N$?

Johann Peter Gustav
Lejeune Dirichlet
(1805/02/13-1859/05/05)



Carl Gottfried
Neumann
(1832/05/07-1925/03/27)

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- Does $\Delta_D \geqslant \Delta_N$?

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- Does $\Delta_D \geqslant \Delta_N$?

- For $\lambda \geqslant 0$, define the Hermitian form

$$G_\lambda(\psi_1, \psi_2) = \int_0^1 \frac{d\bar{\psi}_1}{dx}\frac{d\psi_2}{dx}dx + \lambda\left(\bar{\psi}_1\psi_2|_{x=1} + \bar{\psi}_1\psi_2|_{x=0}\right)$$

- It is obviously that $G_\lambda(\psi, \psi)$ is increasing with λ for generic ψ and nondecreasing for all ψ.

- The operator associated with this Hermitian form is X_λ, which will be also increasing with λ.

- We want to use X_λ representing Δ_D and Δ_N.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- This requires to assign suitable λ to give the "correct" domain for $X_\lambda \psi = \Delta \psi$, which means for all ψ_2 in the domain of G_λ:

$$\langle \Delta \psi_1 | \psi_2 \rangle = G_\lambda(\psi_1, \psi_2) = \int_0^1 \frac{d\bar{\psi}_1}{dx}\frac{d\psi_2}{dx} dx + \lambda \left(\bar{\psi}_1 \psi_2 |_{x=1} + \bar{\psi}_1 \psi_2 |_{x=0} \right)$$

$$= \int_0^1 \left(-\frac{d^2\bar{\psi}_1}{dx^2}\psi_2 \right) dx + \frac{d\bar{\psi}_1}{dx}\psi_2 \bigg|_{x=1} - \frac{d\bar{\psi}_1}{dx}\psi_2 \bigg|_{x=0} + \lambda \left(\bar{\psi}_1 \psi_2 |_{x=1} + \bar{\psi}_1 \psi_2 |_{x=0} \right)$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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- This requires to assign suitable λ to give the "correct" domain for $X_\lambda \psi = \Delta\psi$, which means for all ψ_2 in the domain of G_λ:

$$
\begin{aligned}
\langle \Delta\psi_1 | \psi_2 \rangle = G_\lambda(\psi_1, \psi_2) &= \int_0^1 \frac{d\bar{\psi}_1}{dx}\frac{d\psi_2}{dx} dx + \lambda \left(\bar{\psi}_1 \psi_2 |_{x=1} + \bar{\psi}_1 \psi_2 |_{x=0} \right) \\
&= \int_0^1 \left(-\frac{d^2\bar{\psi}_1}{dx^2}\psi_2 \right) dx + \frac{d\bar{\psi}_1}{dx}\psi_2 \bigg|_{x=1} - \frac{d\bar{\psi}_1}{dx}\psi_2 \bigg|_{x=0} + \lambda \left(\bar{\psi}_1 \psi_2 |_{x=1} + \bar{\psi}_1 \psi_2 |_{x=0} \right)
\end{aligned}
$$

$$
\Rightarrow \quad \begin{cases} \dfrac{d\psi_1}{dx} + \lambda\psi_1 = 0, \; x = 1 \\ \dfrac{d\psi_1}{dx} - \lambda\psi_1 = 0, \; x = 0 \end{cases}
$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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- This requires to assign suitable λ to give the "correct" domain for $X_\lambda \psi = \Delta\psi$, which means for all ψ_2 in the domain of G_λ:

$$\langle \Delta\psi_1 | \psi_2 \rangle = G_\lambda(\psi_1, \psi_2) = \int_0^1 \frac{d\bar{\psi}_1}{dx}\frac{d\psi_2}{dx}dx + \lambda\left(\bar{\psi}_1\psi_2|_{x=1} + \bar{\psi}_1\psi_2|_{x=0}\right)$$

$$= \int_0^1 \left(-\frac{d^2\bar{\psi}_1}{dx^2}\psi_2\right)dx + \frac{d\bar{\psi}_1}{dx}\psi_2\bigg|_{x=1} - \frac{d\bar{\psi}_1}{dx}\psi_2\bigg|_{x=0} + \lambda\left(\bar{\psi}_1\psi_2|_{x=1} + \bar{\psi}_1\psi_2|_{x=0}\right)$$

$$\Rightarrow \begin{cases} \dfrac{d\psi_1}{dx} + \lambda\psi_1 = 0,\ x = 1 \\ \dfrac{d\psi_1}{dx} - \lambda\psi_1 = 0,\ x = 0 \end{cases} \qquad \Rightarrow \begin{cases} \lambda = 0 \ \rightarrow \ \Delta_N \\ \lambda = +\infty \ \rightarrow \ \Delta_D \end{cases}$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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- This requires to assign suitable λ to give the "correct" domain for $X_\lambda \psi = \Delta\psi$, which means for all ψ_2 in the domain of G_λ:

$$\langle \Delta\psi_1 | \psi_2 \rangle = G_\lambda(\psi_1, \psi_2) = \int_0^1 \frac{d\bar{\psi}_1}{dx}\frac{d\psi_2}{dx} dx + \lambda \left(\bar{\psi}_1\psi_2 |_{x=1} + \bar{\psi}_1\psi_2 |_{x=0} \right)$$

$$= \int_0^1 \left(-\frac{d^2\bar{\psi}_1}{dx^2}\psi_2 \right) dx + \frac{d\bar{\psi}_1}{dx}\psi_2 \bigg|_{x=1} - \frac{d\bar{\psi}_1}{dx}\psi_2 \bigg|_{x=0} + \lambda \left(\bar{\psi}_1\psi_2 |_{x=1} + \bar{\psi}_1\psi_2 |_{x=0} \right)$$

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- We have $X_{+\infty} \geqslant X_0$, which is just $\Delta_D \geqslant \Delta_N$.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- $\Delta_D \geqslant \Delta_N$, what does it mean physically?

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- $\Delta_D \geqslant \Delta_N$, what does it mean physically?
- To fix the boundaries of the string on the wall, you have to pay some (probably huge) energy.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

$$X: \ \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

$$X: \ \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m$$

$$X = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}, \quad X_\lambda = \begin{pmatrix} A & B \\ B^\dagger & C + \lambda \end{pmatrix}, \lambda \geqslant 0$$

In Witten's paper, there is a typo in this equation (3.59).

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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$$\frac{1}{s+X} \geqslant \frac{1}{s+X_\lambda}, \quad s \geqslant 0$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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In Witten's paper, there is a typo in this equation (3.59).

$$\frac{1}{s+X} \geqslant \frac{1}{s+X_\lambda}, \quad s \geqslant 0$$

$$\mathbf{1} = (s+X_\lambda)\frac{1}{s+X_\lambda} = \begin{pmatrix} s+A & B \\ B^\dagger & s+C+\lambda \end{pmatrix}\begin{pmatrix} a & b \\ b^\dagger & d \end{pmatrix} = \begin{pmatrix} sa+Aa+Bb^\dagger & sb+Ab+Bd \\ B^\dagger a+(s+C+\lambda)b^\dagger & B^\dagger b+(s+C+\lambda)d \end{pmatrix}$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

  - In the limit $\lambda \to +\infty$

$$\mathbf{1} = \begin{pmatrix} sa + Aa + Bb^\dagger & sb + Ab + Bd \\ B^\dagger a + (s + C + \lambda)b^\dagger & B^\dagger b + (s + C + \lambda)d \end{pmatrix} \to \begin{pmatrix} sa + Aa + Bb^\dagger & sb + Ab + Bd \\ B^\dagger a + \lambda b^\dagger & B^\dagger b + \lambda d \end{pmatrix}$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example
  - In the limit $\lambda \to +\infty$

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$$\Rightarrow \; b^\dagger \sim -\frac{1}{\lambda} B^\dagger a \;\Rightarrow\; \mathbf{1} = \begin{pmatrix} (s + A - BB^\dagger/\lambda) a & -(s + A) a^\dagger B/\lambda + Bd \\ 0 & -B^\dagger a^\dagger B/\lambda + \lambda d \end{pmatrix}$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

  - In the limit $\lambda \to +\infty$

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$$\Rightarrow \; a \sim \frac{1}{s + A} \; \Rightarrow \; \mathbf{1} = \begin{pmatrix} \mathbf{1} & -(s + A)a^\dagger B/\lambda + Bd \\ 0 & -B^\dagger a^\dagger B/\lambda + \lambda d \end{pmatrix} \; \Rightarrow \; d \sim \frac{1}{\lambda}$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

  - In the limit $\lambda \to +\infty$

$$\mathbf{1} = \begin{pmatrix} sa + Aa + Bb^\dagger & sb + Ab + Bd \\ B^\dagger a + (s + C + \lambda)b^\dagger & B^\dagger b + (s + C + \lambda)d \end{pmatrix} \to \begin{pmatrix} sa + Aa + Bb^\dagger & sb + Ab + Bd \\ B^\dagger a + \lambda b^\dagger & B^\dagger b + \lambda d \end{pmatrix}$$

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$$\Rightarrow \ a \sim \frac{1}{s+A} \ \Rightarrow \ \mathbf{1} = \begin{pmatrix} \mathbf{1} & -(s+A)a^\dagger B/\lambda + Bd \\ 0 & -B^\dagger a^\dagger B/\lambda + \lambda d \end{pmatrix} \ \Rightarrow \ d \sim \frac{1}{\lambda}$$

$$\Rightarrow \ \frac{1}{s + X_\lambda} \sim \begin{pmatrix} 1/(s+A) & \mathcal{O}(1/\lambda) \\ \mathcal{O}(1/\lambda) & 1/\lambda \end{pmatrix}$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

$$\frac{1}{s+X} \geqslant \frac{1}{s+X_\lambda}, \quad s \geqslant 0 \;\Rightarrow\; \forall\ \Psi \in \mathbb{C}^{n+m}, \quad \langle \Psi | \frac{1}{s+X} | \Psi \rangle \geqslant \langle \Psi | \frac{1}{s+X_\lambda} | \Psi \rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

$$\frac{1}{s+X} \geqslant \frac{1}{s+X_\lambda}, \quad s \geqslant 0 \;\Rightarrow\; \forall\ \Psi \in \mathbb{C}^{n+m}, \quad \langle\Psi\,|\frac{1}{s+X}|\,\Psi\rangle \geqslant \langle\Psi\,|\frac{1}{s+X_\lambda}|\,\Psi\rangle$$

For $\Psi = \begin{pmatrix}\psi\\0\end{pmatrix}, \lambda \to +\infty \;\Rightarrow\; \langle\Psi\,|\frac{1}{s+X_\lambda}|\,\Psi\rangle \to \langle\psi\,|\,(s+A)^{-1}\,|\,\psi\rangle$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

$$\frac{1}{s+X} \geqslant \frac{1}{s+X_\lambda}, \quad s \geqslant 0 \;\Rightarrow\; \forall\ \Psi \in \mathbb{C}^{n+m}, \quad \langle \Psi | \frac{1}{s+X} | \Psi \rangle \geqslant \langle \Psi | \frac{1}{s+X_\lambda} | \Psi \rangle$$

For $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \lambda \to +\infty \;\Rightarrow\; \langle \Psi | \dfrac{1}{s+X_\lambda} | \Psi \rangle \to \langle \psi | (s+A)^{-1} | \psi \rangle$

Define an isometric embedding $U : \mathbb{C}^n \to \mathbb{C}^{n+m}$ by $U(\psi) = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

$$\frac{1}{s+X} \geqslant \frac{1}{s+X_\lambda}, \quad s \geqslant 0 \;\Rightarrow\; \forall\ \Psi \in \mathbb{C}^{n+m}, \quad \langle \Psi | \frac{1}{s+X} | \Psi \rangle \geqslant \langle \Psi | \frac{1}{s+X_\lambda} | \Psi \rangle$$

For $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$, $\lambda \to +\infty \;\Rightarrow\; \langle \Psi | \frac{1}{s+X_\lambda} | \Psi \rangle \to \langle \psi | (s+A)^{-1} | \psi \rangle$

Define an isometric embedding $U : \mathbb{C}^n \to \mathbb{C}^{n+m}$ by $U(\psi) = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$

$$\langle \Psi | \frac{1}{s+X} | \Psi \rangle = \langle \psi | U^\dagger \frac{1}{s+X} U | \psi \rangle \geqslant \langle \psi | \frac{1}{s+A} | \psi \rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

V. Example

- A finite dimensional example

$$\frac{1}{s+X} \geqslant \frac{1}{s+X_\lambda}, \quad s \geqslant 0 \;\Rightarrow\; \forall\ \Psi \in \mathbb{C}^{n+m}, \quad \langle \Psi | \frac{1}{s+X} | \Psi \rangle \geqslant \langle \Psi | \frac{1}{s+X_\lambda} | \Psi \rangle$$

For $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$, $\lambda \to +\infty \;\Rightarrow\; \langle \Psi | \frac{1}{s+X_\lambda} | \Psi \rangle \to \langle \psi | (s+A)^{-1} | \psi \rangle$

Define an isometric embedding $U : \mathbb{C}^n \to \mathbb{C}^{n+m}$ by $U(\psi) = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$

$$\langle \Psi | \frac{1}{s+X} | \Psi \rangle = \langle \psi | U^\dagger \frac{1}{s+X} U | \psi \rangle \geqslant \langle \psi | \frac{1}{s+A} | \psi \rangle$$

$$\Rightarrow \quad \langle \psi | U^\dagger (\log X) U | \psi \rangle \leqslant \langle \psi | \log A | \psi \rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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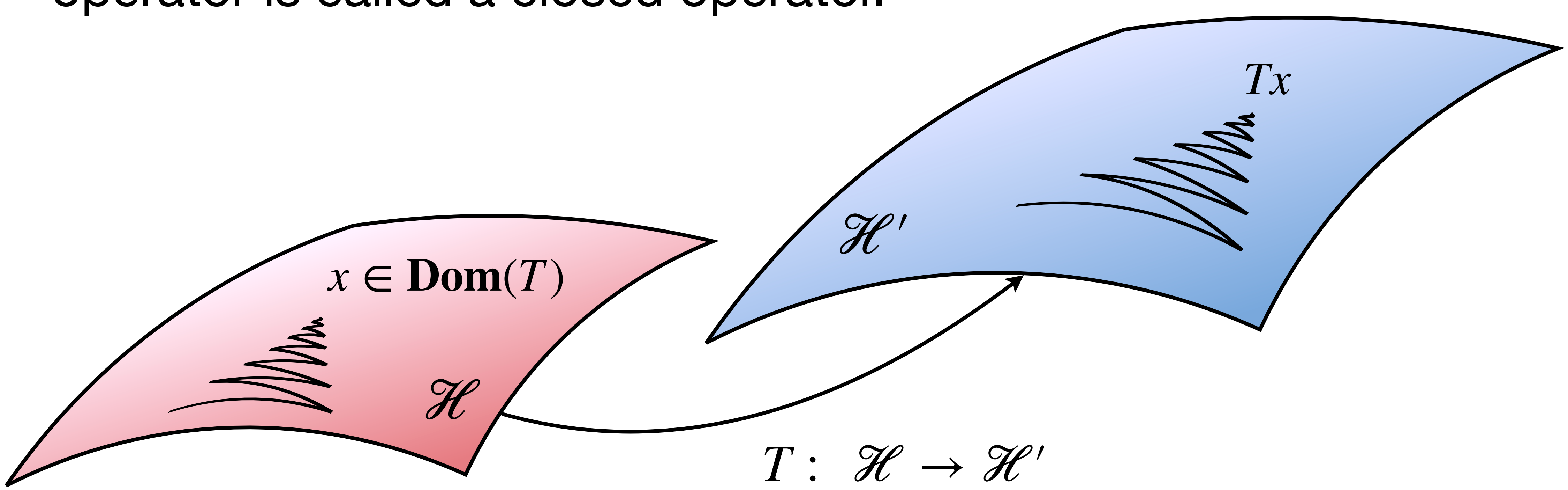
THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

VI. The proof

- Closed operator and its Graph

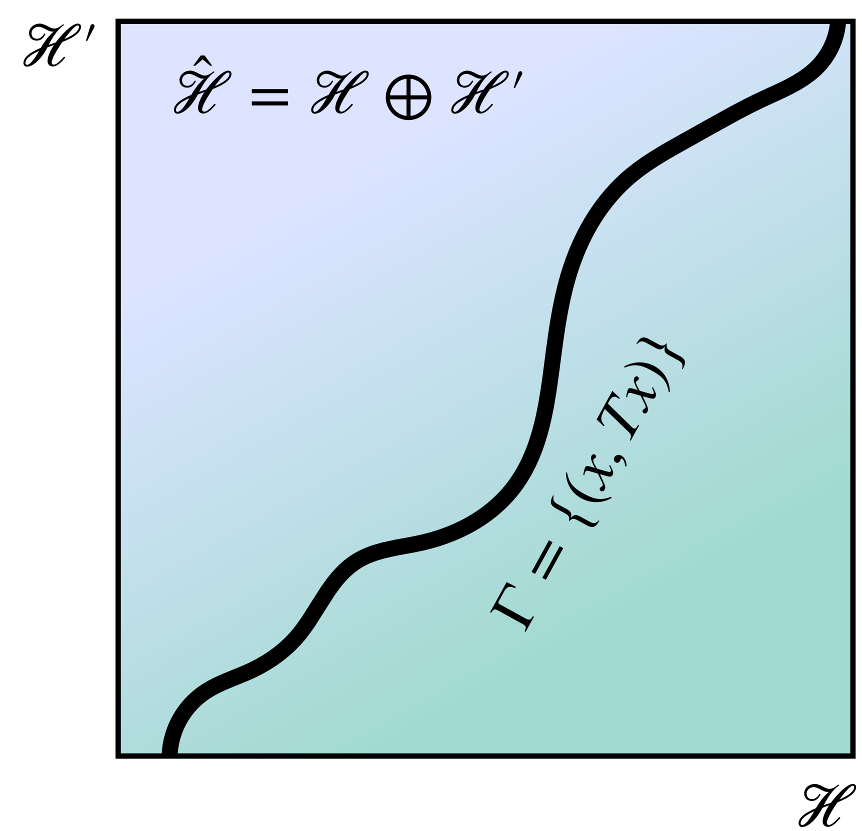
  - **Closed operator**: for an unbounded operator $T: \ \mathscr{H} \to \mathscr{H}'$, if for any sequence $\{x_n\}$ in its domain, the existence of the limits $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} Tx_n = y$ ensures $x \in \mathbf{Dom}(T)$ and $Tx = y$, then the operator is called a closed operator.



THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

VI. The proof

- Closed operator and its Graph
  - **Graph**: the set $\Gamma = \{(x, Tx) \mid x \in \mathbf{Dom}(T)\}$ in $\hat{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}'$ is called the graph of the operator T.
  - T is closed operator $\Leftrightarrow$ Γ is closed subset of $\hat{\mathscr{H}}$



THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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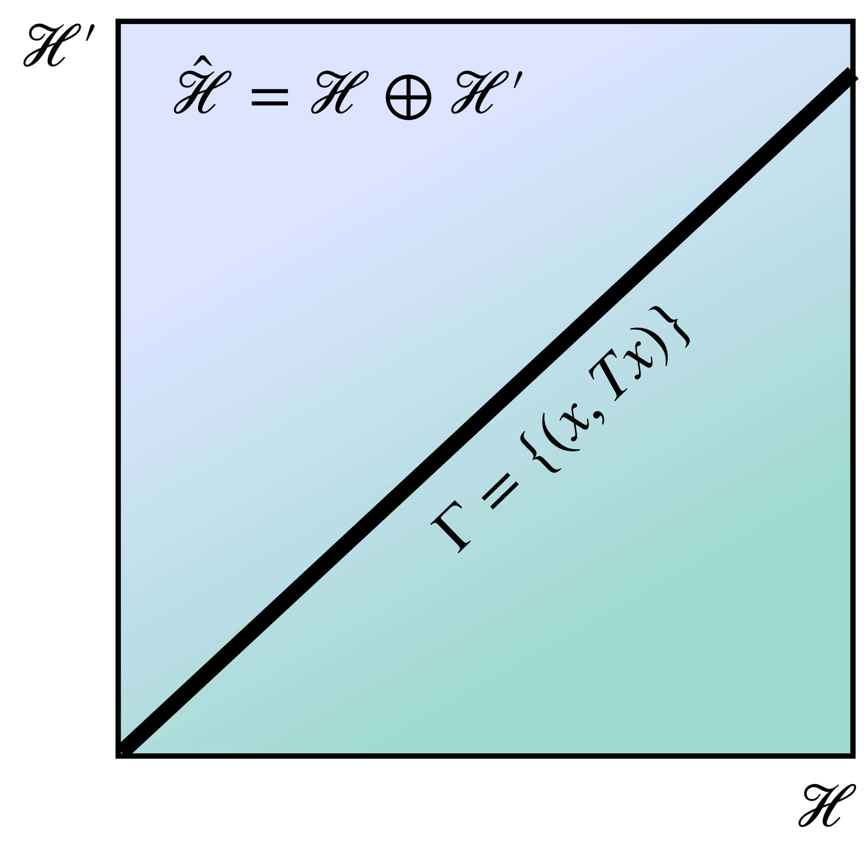
  - **Graph**: the set $\Gamma = \{(x, Tx) \mid x \in \mathbf{Dom}(T)\}$ in $\hat{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}'$ is called the graph of the operator T.
  - T is closed linear operator $\Leftrightarrow$ Γ is Hilbert subspace of $\hat{\mathscr{H}}$



THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

VI. The proof

- The orthogonal projector Π of Γ

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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- The orthogonal projector Π of Γ

$$\psi \in \mathscr{H}, \chi \in \mathscr{H}', \Pi \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} x \\ Tx \end{pmatrix} \text{ and } \begin{pmatrix} \psi - x \\ \varphi - Tx \end{pmatrix} \perp \begin{pmatrix} x \\ Tx \end{pmatrix}$$

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$$\Rightarrow \ \langle x | \psi - x \rangle + \langle Tx | \varphi - Tx \rangle = 0 \ \Rightarrow \ \langle x | \psi \rangle + \langle x | T^\dagger \varphi \rangle = \langle x | x \rangle + \langle x | T^\dagger Tx \rangle$$

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$$\Rightarrow \ \langle x | \psi + T^\dagger \varphi \rangle = \langle x | (1 + T^\dagger T) | x \rangle \ \Rightarrow \ | x \rangle = (1 + T^\dagger T)^{-1} (| \psi \rangle + T^\dagger | \varphi \rangle)$$

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$$\Rightarrow \Pi = \begin{pmatrix} (1 + T^\dagger T)^{-1} & (1 + T^\dagger T)^{-1} T^\dagger \\ T(1 + T^\dagger T)^{-1} & T(1 + T^\dagger T)^{-1} T^\dagger \end{pmatrix}$$

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- The orthogonal projector Π of Γ

  $\psi \in \mathscr{H}, \chi \in \mathscr{H}', \Pi \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} x \\ Tx \end{pmatrix}$ and $\begin{pmatrix} \psi - x \\ \varphi - Tx \end{pmatrix} \perp \begin{pmatrix} x \\ Tx \end{pmatrix}$

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- The orthogonal projector Π and $(1 + T^\dagger T)^{-1}$, $(1 + T^\dagger T)^{-1} T^\dagger$, $T(1 + T^\dagger T)^{-1}$, $T(1 + T^\dagger T)^{-1} T^\dagger$ are all **bounded** operators so they can be defined on the whole $\hat{\mathscr{H}}$.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

VI. The proof

- The proof of the monotonicity of relative entropy
  - Let T_0 and T_1 are two closed operators and $T_0 \subset T_1$, then $\Gamma_0 \subseteq \Gamma_1$ and it is obviously that $\Pi_1 \geqslant \Pi_0$ ($\forall\ \Psi$, $\langle \Psi | \Pi_1 | \Psi \rangle \geqslant \langle \Psi | \Pi_0 | \Psi \rangle$).
  - For vectors $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ and operators $T_{0,1}/\sqrt{s}$

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  - For vectors $\Psi = \binom{\psi}{0}$ and operators $T_{0,1}/\sqrt{s}$

$$\langle \psi | \frac{1}{s + T_0^\dagger T_0} | \psi \rangle \leqslant \langle \psi | \frac{1}{s + T_1^\dagger T_1} | \psi \rangle$$

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  - For vectors $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ and operators $T_{0,1}/\sqrt{s}$

$$\langle \psi | \frac{1}{s + T_0^\dagger T_0} | \psi \rangle \leqslant \langle \psi | \frac{1}{s + T_1^\dagger T_1} | \psi \rangle$$

$$\Rightarrow \quad T_1^\dagger T_1 \leqslant T_0^\dagger T_0, \quad \log T_1^\dagger T_1 \leqslant \log T_0^\dagger T_0$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

VI. The proof

- Some examples:
  - Linear unbounded operator T, $\lim_{n\to\infty} x_n = x$ but $\lim_{n\to\infty} Tx_n$ does not exist

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$$\mathscr{H} = L^2_{\mathbb{C}}(0,\ 1),\quad T : f(x) \mapsto \frac{df}{dx},\quad f_n(x) = \frac{\sqrt{2}}{n}\sin(n^2\pi x)$$

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$$Tf_n = \sqrt{2}n\cos(n^2\pi x)\ \ \Rightarrow\ \ \|Tf_n\| = n,\ \ \therefore\ \ Tf_n \in L^2_{\mathbb{C}}(0,\ 1),\ \ f_n \in \mathbf{Dom}(T)$$

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But it is obviously that the limit $\lim_{n\to\infty} Tf_n$ does not exist.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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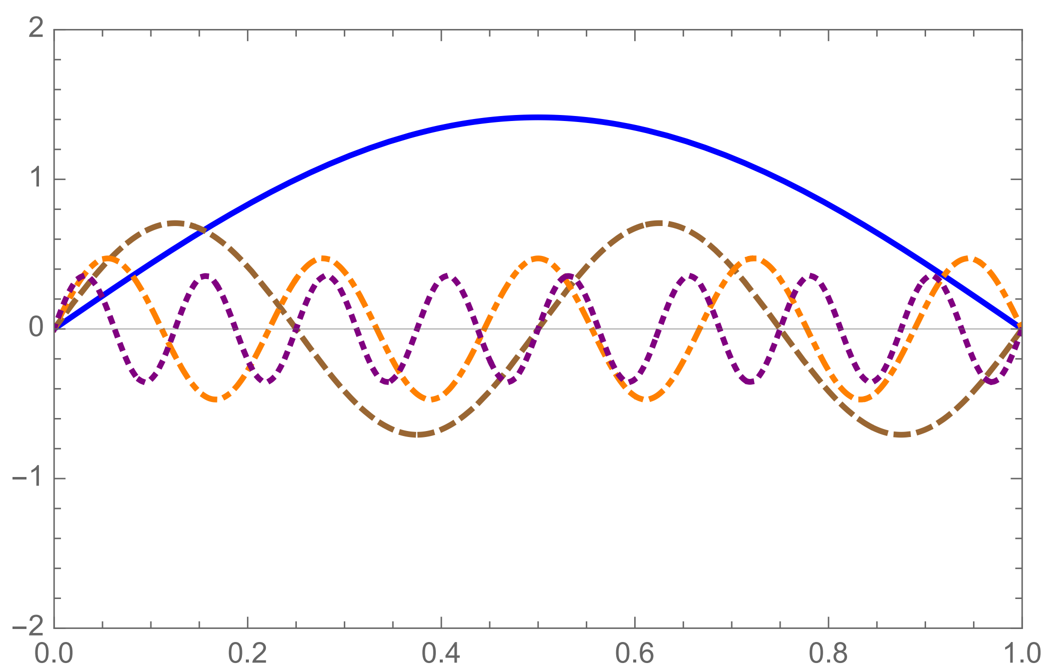

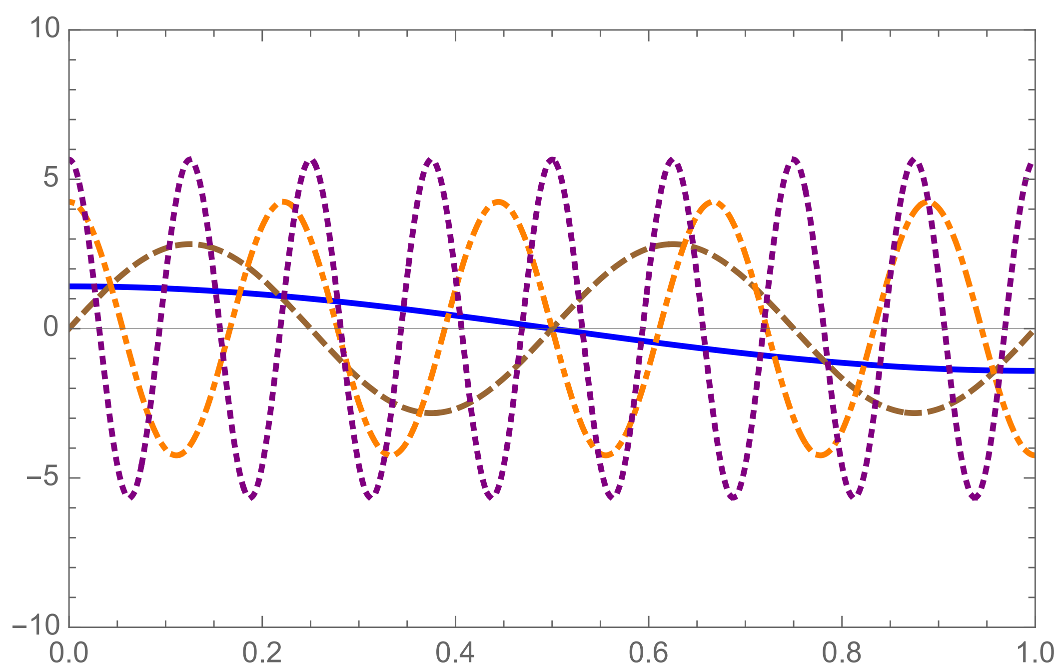

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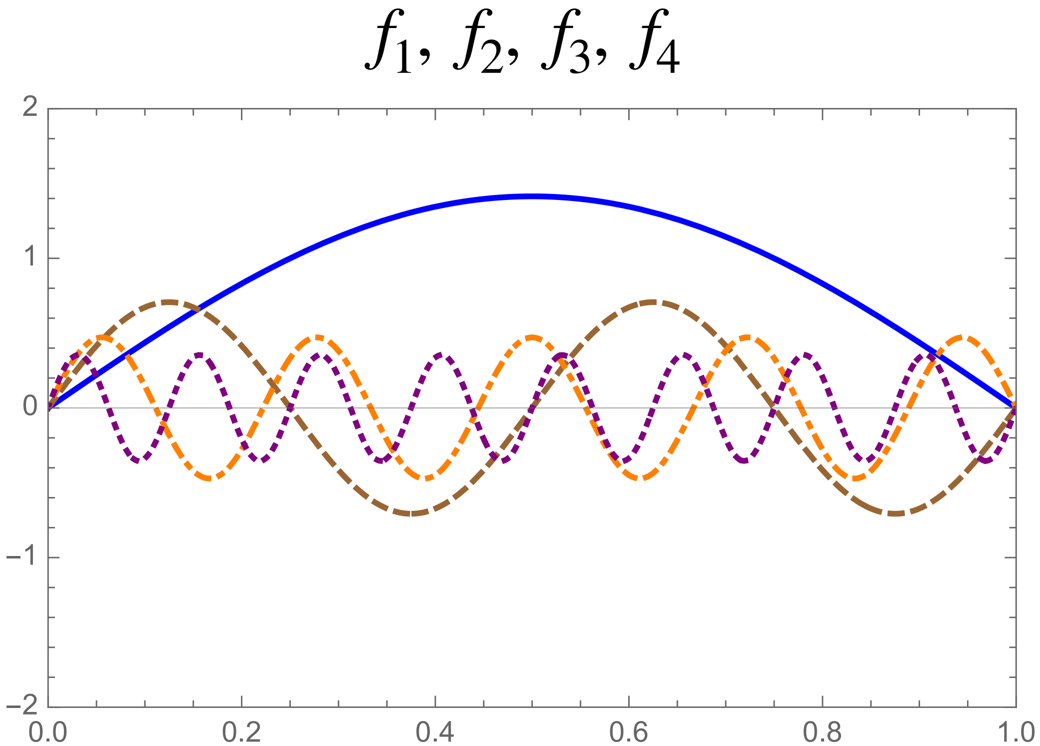

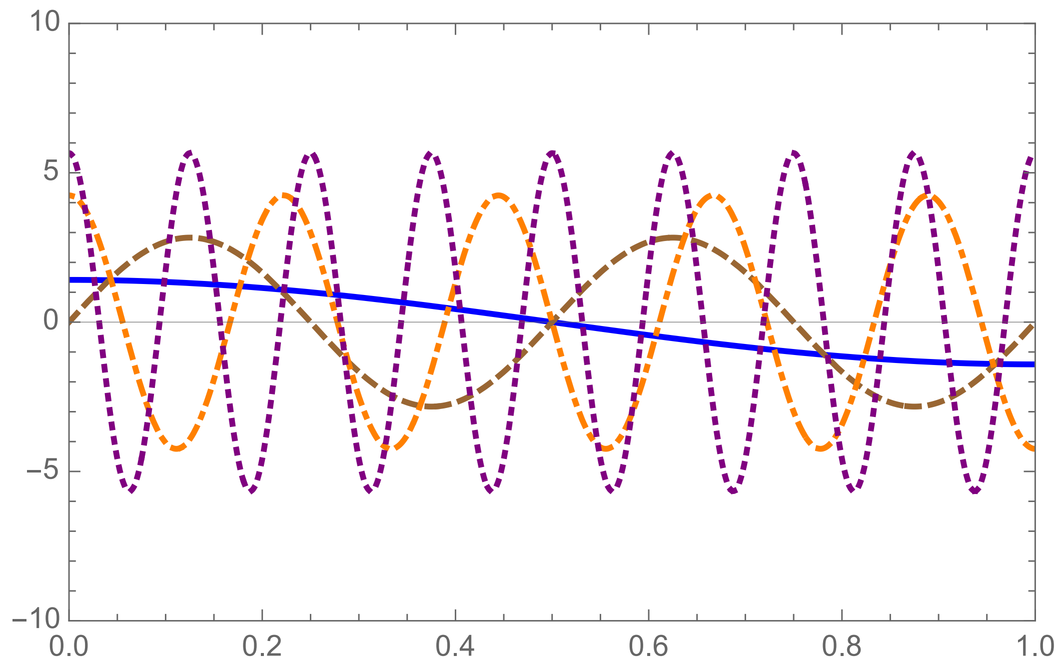

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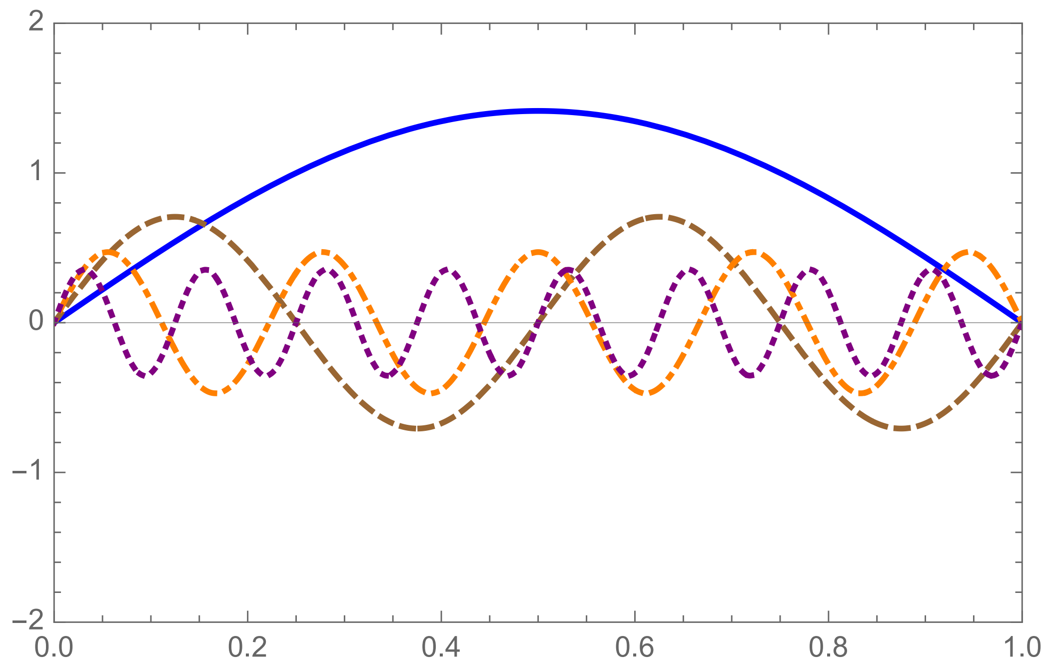
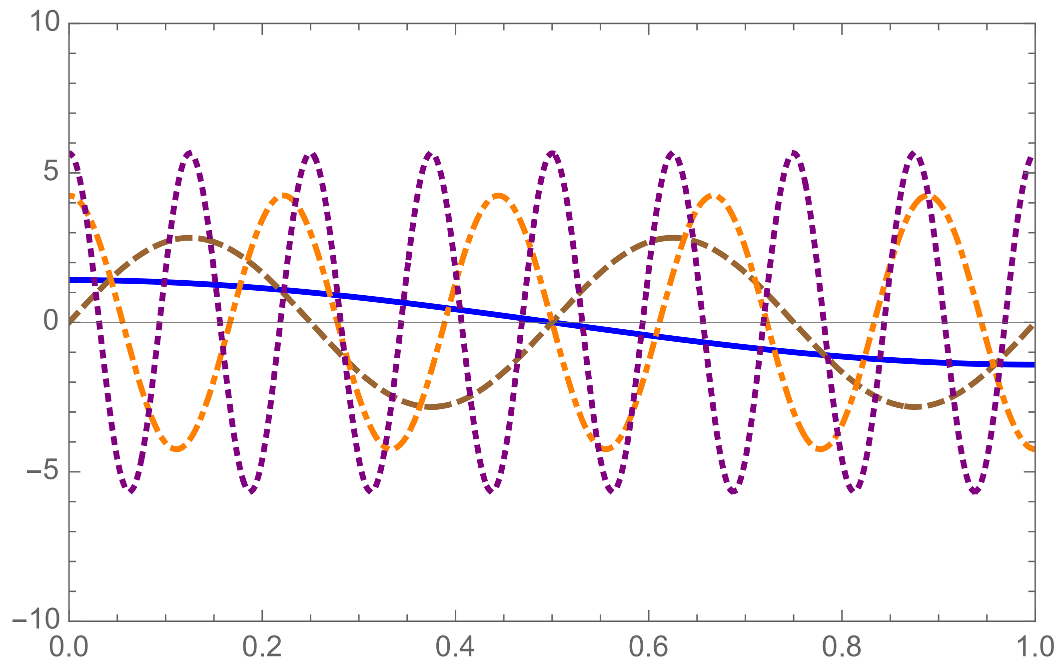
f_1, f_2, f_3, f_4



Tf_1, Tf_2, Tf_3, Tf_4



THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

VI. The proof

- Some examples:

  - $\Gamma_0 \subsetneq \Gamma_1$: consider the 1-dim wave function with different boundary conditions again.

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$$\mathscr{H} = L^2_{\mathbb{C}}(0,\,1),\;\; T_0 : f(x) \mapsto \frac{df}{dx},\;\; \mathbf{Dom}(T_0) = \{f\,|\,f(0) = f(1) = 0,\, f \in \mathscr{C}^1(0,\,1),\, f' \in \mathscr{H}\}$$

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$$(f, f') \in \Gamma_0,\ (g, g') \in \Gamma_1,\ \ \Gamma_{0,1} \subset L^2_{\mathbb{C}}(0,\ 1) \oplus L^2_{\mathbb{C}}(0,\ 1)$$

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$$(f, f') \perp (g, g') \in \Gamma_1 \;\Rightarrow\; \int_0^1 dx\,\bar{f}g + \int_0^1 dx \frac{d\bar{f}}{dx}\frac{dg}{dx} = 0$$

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$$\int_0^1 dx\, \bar{f} g + \int_0^1 dx \frac{d\bar{f}}{dx}\frac{dg}{dx} = 0 \;\Rightarrow\; \int_0^1 dx\, \bar{f}\left(g - \frac{d^2 g}{dx^2}\right) + \bar{f}\frac{dg}{dx}\Big|_0^1 = 0$$

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Γ_0 is of codimension two in Γ_1.


To Be Continued...