Entanglement properties of quantum field theory

A note of Witten's paper "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory"

Part III: From Finite-dimensional Quantum Systems and Some Lessons to A Fundamental Example in Quantum Field Theory

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FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The "representation matrices" of modular operators

- The cyclic and separating vector

$$\Psi = \mathrm{tr}\left[\begin{pmatrix} |c_1| & 0 & \cdots & 0 \\ 0 & |c_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |c_n| \end{pmatrix}\begin{pmatrix} |1,1\rangle & |2,1\rangle & \cdots & |n,1\rangle \\ |1,2\rangle & |2,2\rangle & \cdots & |n,2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |1,n\rangle & |2,n\rangle & \cdots & |n,n\rangle \end{pmatrix}\right]$$

$$C_\Psi = \rho_1^{1/2}$$

- Although $\hat{\rho}_1 \neq \hat{\rho}_2$, the "representation matrices" $\rho_1 = \rho_2$.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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$$\Delta_\Psi |i, j\rangle = |c_i/c_j|^2 |i, j\rangle$$

$$\Delta_\Psi \Xi = \sum_{i,j=1}^{n} |c_i|^2 c_{ij} |c_j|^{-2} |i, j\rangle, \quad \Rightarrow \quad C_\Xi \to \rho_1 C_\Xi \rho_2^{-1}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

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$$\Delta_{\Psi|\Phi}(C_X) = \sigma_1 C_X \rho_2^{-1} = \sigma_1 C_X \rho_1^{-1}$$

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$$\Delta_{\Psi|\Phi}(C_X) = \sigma_1 C_X \rho_2^{-1} = \sigma_1 C_X \rho_1^{-1}$$

$$\therefore \ \ \Delta_{\Psi|\Phi}^{\alpha}(C_X) = \sigma_1^{\alpha} C_X \rho_1^{-\alpha}$$

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$$\Delta_{\Psi|\Phi}(C_X) = \sigma_1 C_X \rho_2^{-1} = \sigma_1 C_X \rho_1^{-1}$$

$$\therefore \;\; \Delta^{\alpha}_{\Psi|\Phi}(C_X) = \sigma_1^{\alpha} C_X \rho_1^{-\alpha}$$

$$\Rightarrow \;\; \langle \Psi \,|\, \Delta^{\alpha}_{\Psi|\Phi} \,|\, \Psi \rangle = \mathrm{tr}\left[\rho_1^{1/2}\; \Delta^{\alpha}_{\Psi|\Phi}(C_\Psi)\right]$$

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$$= \mathrm{tr}\left[\rho_1^{1/2} \; \left(\sigma_1^{\alpha} \rho_1^{1/2} \rho_1^{-\alpha}\right)\right]$$

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$$\therefore \ \Delta^{\alpha}_{\Psi|\Phi}(C_X) = \sigma_1^{\alpha} C_X \rho_1^{-\alpha}$$

$$\begin{aligned}
\Rightarrow \ \langle \Psi | \Delta^{\alpha}_{\Psi|\Phi} | \Psi \rangle &= \mathrm{tr}\left[\rho_1^{1/2} \ \Delta^{\alpha}_{\Psi|\Phi}(C_\Psi)\right] \\
&= \mathrm{tr}\left[\rho_1^{1/2} \ \left(\sigma_1^{\alpha} \rho_1^{1/2} \rho_1^{-\alpha}\right)\right] \\
&= \mathrm{tr}\left[\sigma_1^{\alpha} \rho_1^{1-\alpha}\right]
\end{aligned}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The "representation matrices" of modular operators

- Because the bases are fixed by the "diagonalization" of the Ψ, but not Φ, one usually does not have simple relations such as $\sigma_1 = \sigma_2$.

$$\Phi = \mathrm{tr}\left[\begin{pmatrix} \langle 1, 1|\Phi\rangle & \langle 1, 2|\Phi\rangle & \cdots & \langle 1, n|\Phi\rangle \\ \langle 2, 1|\Phi\rangle & \langle 2, 2|\Phi\rangle & \cdots & \langle 2, n|\Phi\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n, 1|\Phi\rangle & \langle n, 2|\Phi\rangle & \cdots & \langle n, n|\Phi\rangle \end{pmatrix}\begin{pmatrix} |1, 1\rangle & |2, 1\rangle & \cdots & |n, 1\rangle \\ |1, 2\rangle & |2, 2\rangle & \cdots & |n, 2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |1, n\rangle & |2, n\rangle & \cdots & |n, n\rangle \end{pmatrix}\right]$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- If we are only interested in $\hat{\sigma}_1$ and not $\hat{\sigma}_2$, we can make any unitary transformation on $\mathscr{H}_2$.

- For example, the unitary transformation: $U : \{|\tilde{\varphi}_\alpha\rangle\} \to \{|\varphi_i\rangle\}$.

- On the other hand, by polar decomposition theorem, one has $\Phi = PU$, where P is a positive Hermitian matrix and U is a unitary matrix which acts on $\mathscr{H}_2$.

- It is obviously that $P = \sigma_1^{1/2}$. So with a unitary transformation on $\mathscr{H}_2$, one has $\Phi = \sigma_1^{1/2}$.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- Stone theorem (1930) and 1-parameter automorphism group:

A self-adjoint operator A defined on some dense subset of the Hilbert space

⟺

A strong continued 1-parameter unitary transformation group $U(t)=\exp(itA)$

Marshall Harvey Stone
(1903/04/08-1989/01/09)

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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- Stone theorem (1930) and 1-parameter automorphism group

- The modular automorphism group: the self-adjoint modular operator Δ_Ψ generates a 1-parameter unitary transformation group by

$$\Delta_\Psi^{is}, \quad s \in \mathbb{R}$$

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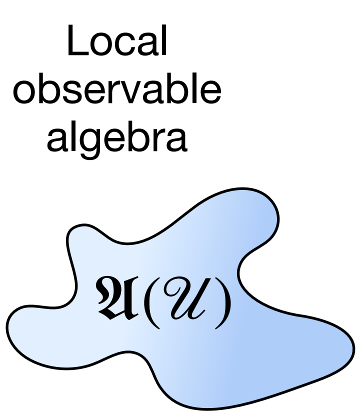

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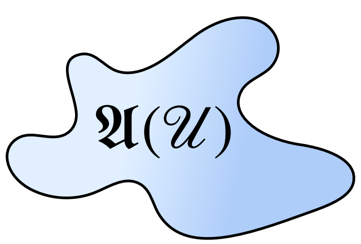
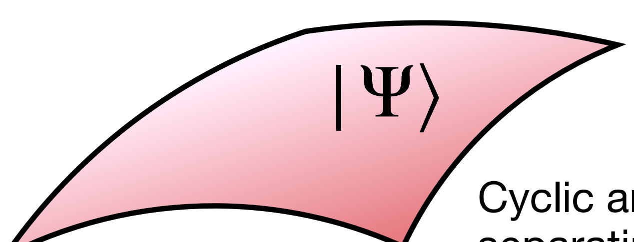
Local observable algebra





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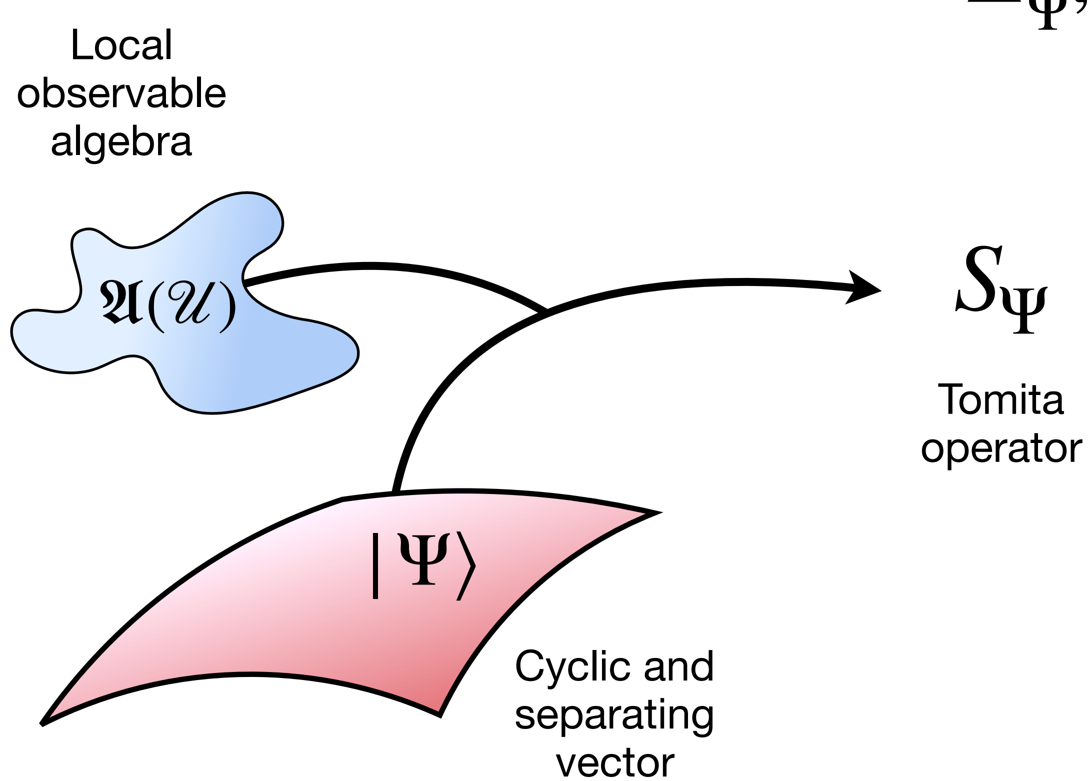

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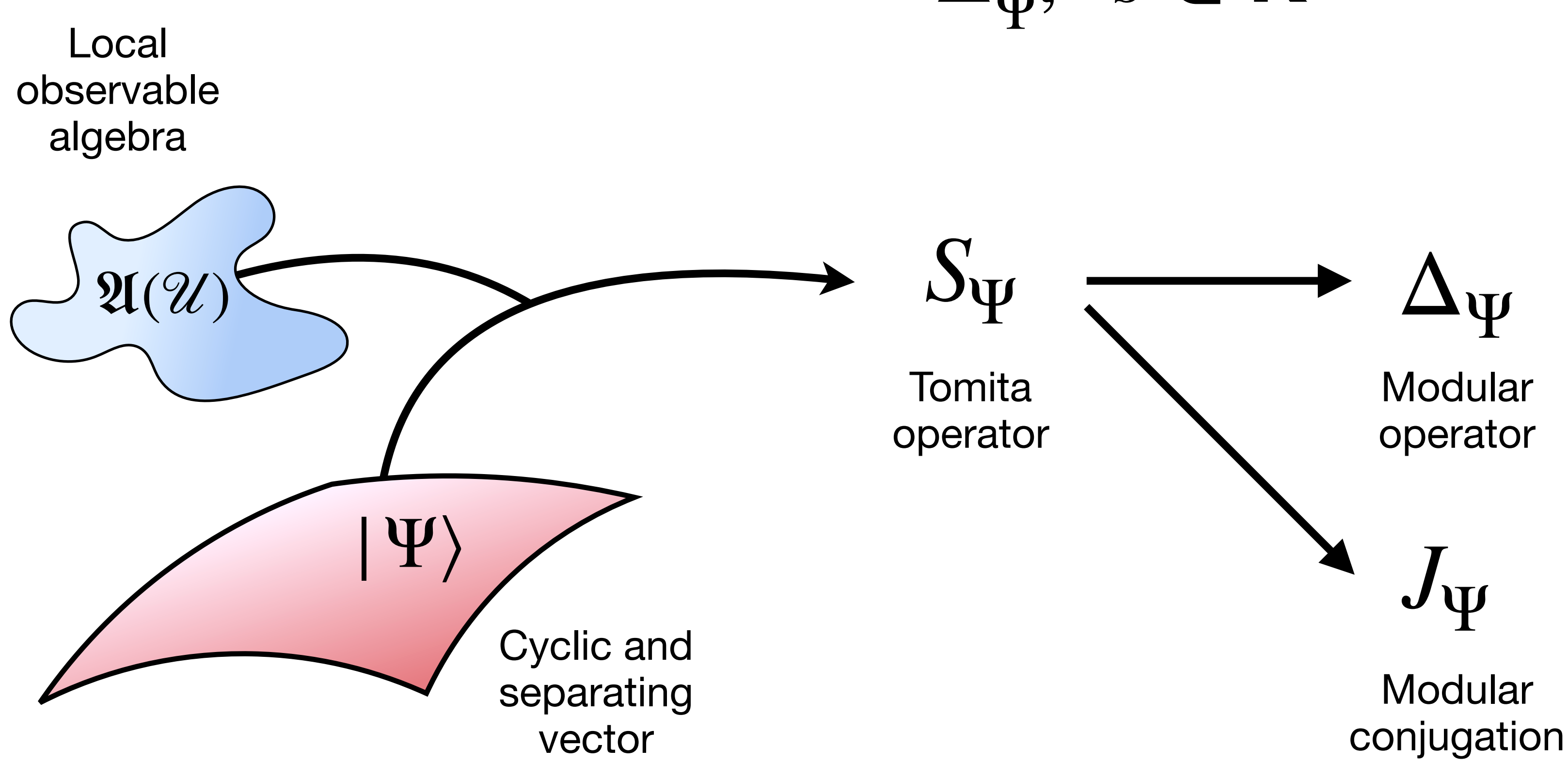

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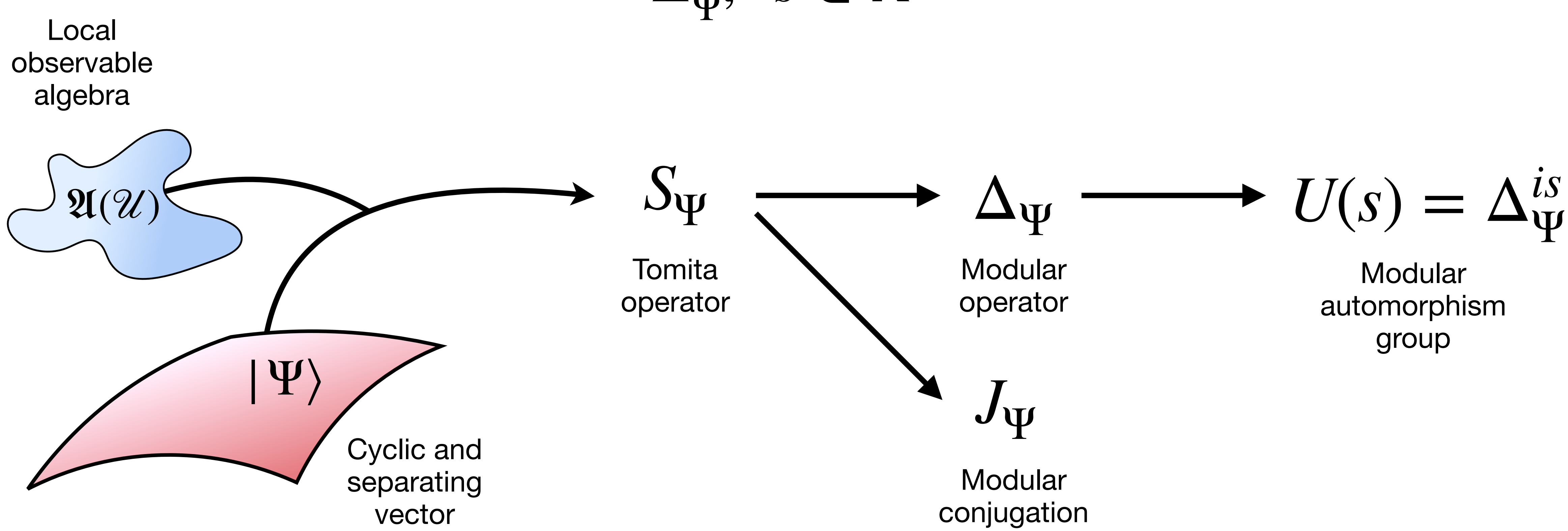


FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

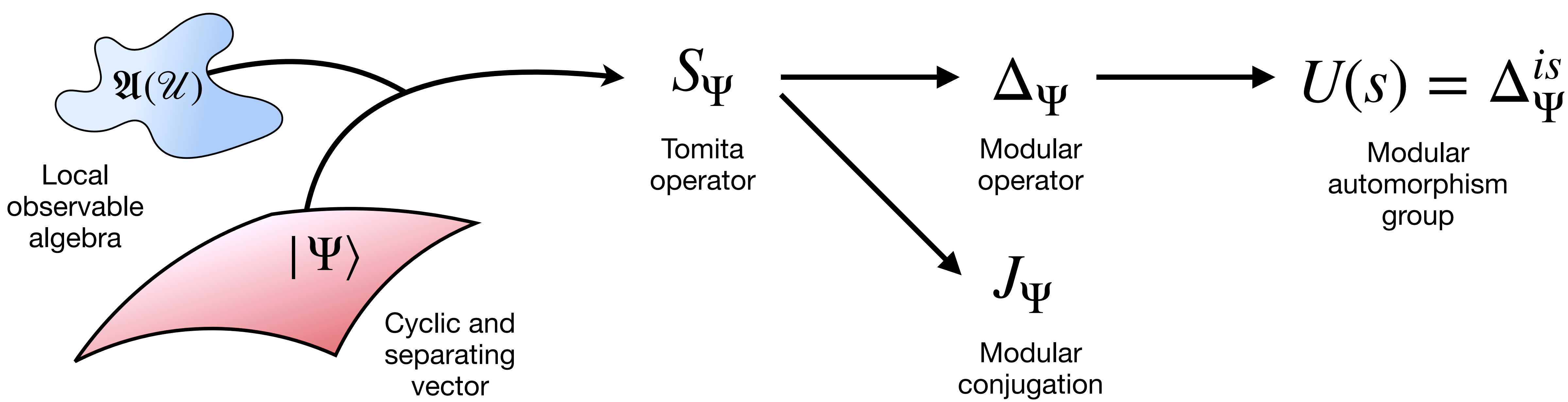
II. The modular automorphism group

- The properties of the modular automorphism group

  1. Δ_Ψ^{is} commutes with J_Ψ;

  2. Since $\Delta_\Psi^{is} = \rho_1^{is} \otimes \rho_2^{-is}$, for any $\mathbf{a} \otimes \mathbf{1} \in \mathfrak{A}$,

$$\Delta_\Psi^{is}(\mathbf{a} \otimes \mathbf{1})\Delta_\Psi^{-is} = \rho_1^{is}\mathbf{a}\rho_1^{-is} \otimes \mathbf{1}$$



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The properties of the modular automorphism group

  1. $J_\Psi \Delta_\Psi^{is} J_\Psi = \Delta_\Psi^{is}$;

  2. $\Delta_\Psi^{is}(\mathbf{a} \otimes \mathbf{1})\Delta_\Psi^{-is} = \rho_1^{is}\mathbf{a}\rho_1^{-is} \otimes \mathbf{1}$;

  3. $\Delta_\Psi^{is}\ \mathfrak{A}\ \Delta_\Psi^{-is} = \mathfrak{A},\ \ \Delta_\Psi^{is}\ \mathfrak{A}'\ \Delta_\Psi^{-is} = \mathfrak{A}'$;

  4. $J_\Psi \mathfrak{A} J_\Psi = \mathfrak{A}',\ \ J_\Psi \mathfrak{A}' J_\Psi = \mathfrak{A}$;

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3. $\Delta_\Psi^{is}\ \mathfrak{A}\ \Delta_\Psi^{-is} = \mathfrak{A},\ \ \Delta_\Psi^{is}\ \mathfrak{A}'\ \Delta_\Psi^{-is} = \mathfrak{A}'$;

4. $J_\Psi \mathfrak{A} J_\Psi = \mathfrak{A}',\ \ J_\Psi \mathfrak{A}' J_\Psi = \mathfrak{A}$;

$$J_\Psi(\mathbf{a} \otimes \mathbf{1})J_\Psi |i, j\rangle = J_\Psi(\mathbf{a} \otimes \mathbf{1}) |j, i\rangle = \sum_k J_\Psi a_{kj} |k, i\rangle = \sum_k \bar{a}_{kj} J_\Psi |k, i\rangle$$
$$= \sum_k \bar{a}_{kj} |i, k\rangle = (\mathbf{1} \otimes \mathbf{a}^*) |i, j\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The properties of the modular automorphism group

1. $J_\Psi \Delta_\Psi^{is} J_\Psi = \Delta_\Psi^{is}$;
2. $\Delta_\Psi^{is}(\mathbf{a} \otimes \mathbf{1})\Delta_\Psi^{-is} = \rho_1^{is}\mathbf{a}\rho_1^{-is} \otimes \mathbf{1}$;
3. $\Delta_\Psi^{is}\ \mathfrak{A}\ \Delta_\Psi^{-is} = \mathfrak{A},\ \ \Delta_\Psi^{is}\ \mathfrak{A}'\ \Delta_\Psi^{-is} = \mathfrak{A}'$;
4. $J_\Psi \mathfrak{A} J_\Psi = \mathfrak{A}',\ \ J_\Psi \mathfrak{A}' J_\Psi = \mathfrak{A}$;
5. $J_\Psi(\mathbf{a} \otimes \mathbf{1})J_\Psi = \mathbf{1} \otimes \mathbf{a}^*,\ \ J_\Psi(\mathbf{1} \otimes \mathbf{a})J_\Psi = \mathbf{a}^* \otimes \mathbf{1}$;

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The group generated by relative modular operator is called "relative modular group"

$$\Delta_{\Psi|\Phi}^{is}(\mathbf{a}\otimes\mathbf{1})\Delta_{\Psi|\Phi}^{-is} = \sigma_1^{is}\mathbf{a}\sigma_1^{-is}\otimes\mathbf{1}$$

- The relative modular group also has properties

1. $\Delta_{\Psi|\Phi}^{is}\ \mathfrak{A}\ \Delta_{\Psi|\Phi}^{-is} = \mathfrak{A},\ \ \Delta_{\Psi|\Phi}^{is}\ \mathfrak{A}'\ \Delta_{\Psi|\Phi}^{-is} = \mathfrak{A}';$
2. $J_{\Psi|\Phi}\mathfrak{A}J_{\Psi|\Phi} = \mathfrak{A}',\ \ J_{\Psi|\Phi}\mathfrak{A}'J_{\Psi|\Phi} = \mathfrak{A};$
3. $J_{\Psi|\Phi}(\mathbf{a}\otimes\mathbf{1})J_{\Psi|\Phi} = \mathbf{1}\otimes\mathbf{a}^*,\ \ J_{\Psi|\Phi}(\mathbf{1}\otimes\mathbf{a})J_{\Psi|\Phi} = \mathbf{a}^*\otimes\mathbf{1};$

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- The relative modular group also has properties

  1. $\Delta^{is}_{\Psi|\Phi}\ \mathfrak{A}\ \Delta^{-is}_{\Psi|\Phi} = \mathfrak{A},\ \ \Delta^{is}_{\Psi|\Phi}\ \mathfrak{A}'\ \Delta^{-is}_{\Psi|\Phi} = \mathfrak{A}';$

  2. $J_{\Psi|\Phi}\mathfrak{A}J_{\Psi|\Phi} = \mathfrak{A}',\ \ J_{\Psi|\Phi}\mathfrak{A}'J_{\Psi|\Phi} = \mathfrak{A};$

  3. $J_{\Psi|\Phi}(\mathbf{a} \otimes \mathbf{1})J_{\Psi|\Phi} = \mathbf{1} \otimes \mathbf{a}^*,\ \ J_{\Psi|\Phi}(\mathbf{1} \otimes \mathbf{a})J_{\Psi|\Phi} = \mathbf{a}^* \otimes \mathbf{1}.$

- And $\Delta^{is}_{\Psi|\Phi}(\mathbf{a} \otimes \mathbf{1})\Delta^{-is}_{\Psi|\Phi} = \Delta^{is}_{\Psi'|\Phi}(\mathbf{a} \otimes \mathbf{1})\Delta^{-is}_{\Psi'|\Phi}$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- These properties are main theorems of Tomita-Takesaki theory
- The theorems are also true for general infinite-dimensional von Neumann algebras with cyclic separating vectors
- They are not easy to prove

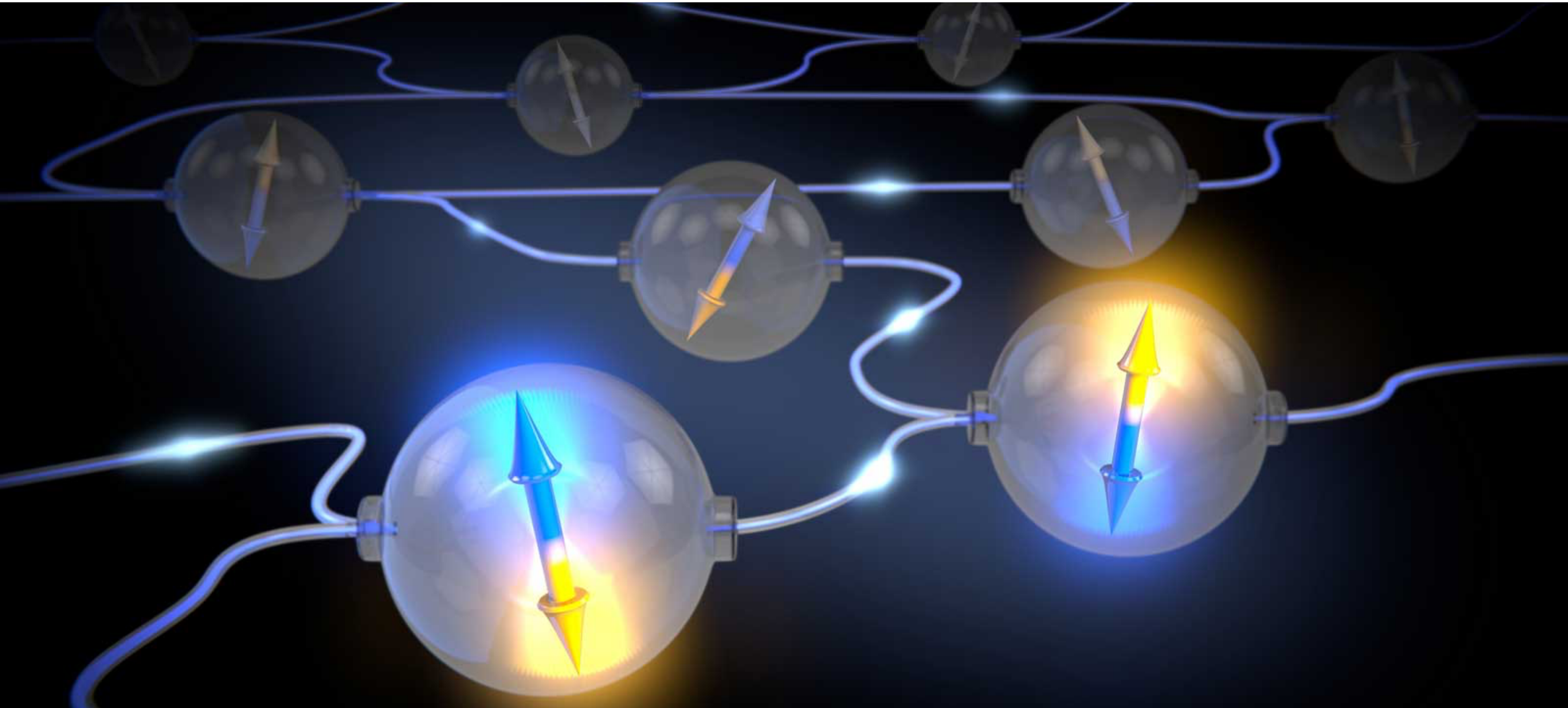
Minoru Tomita
冨田 稔
(1924/02/06-2015/10/09)

Masamichi Takesaki
竹崎 正道
(1933/07/18-)

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

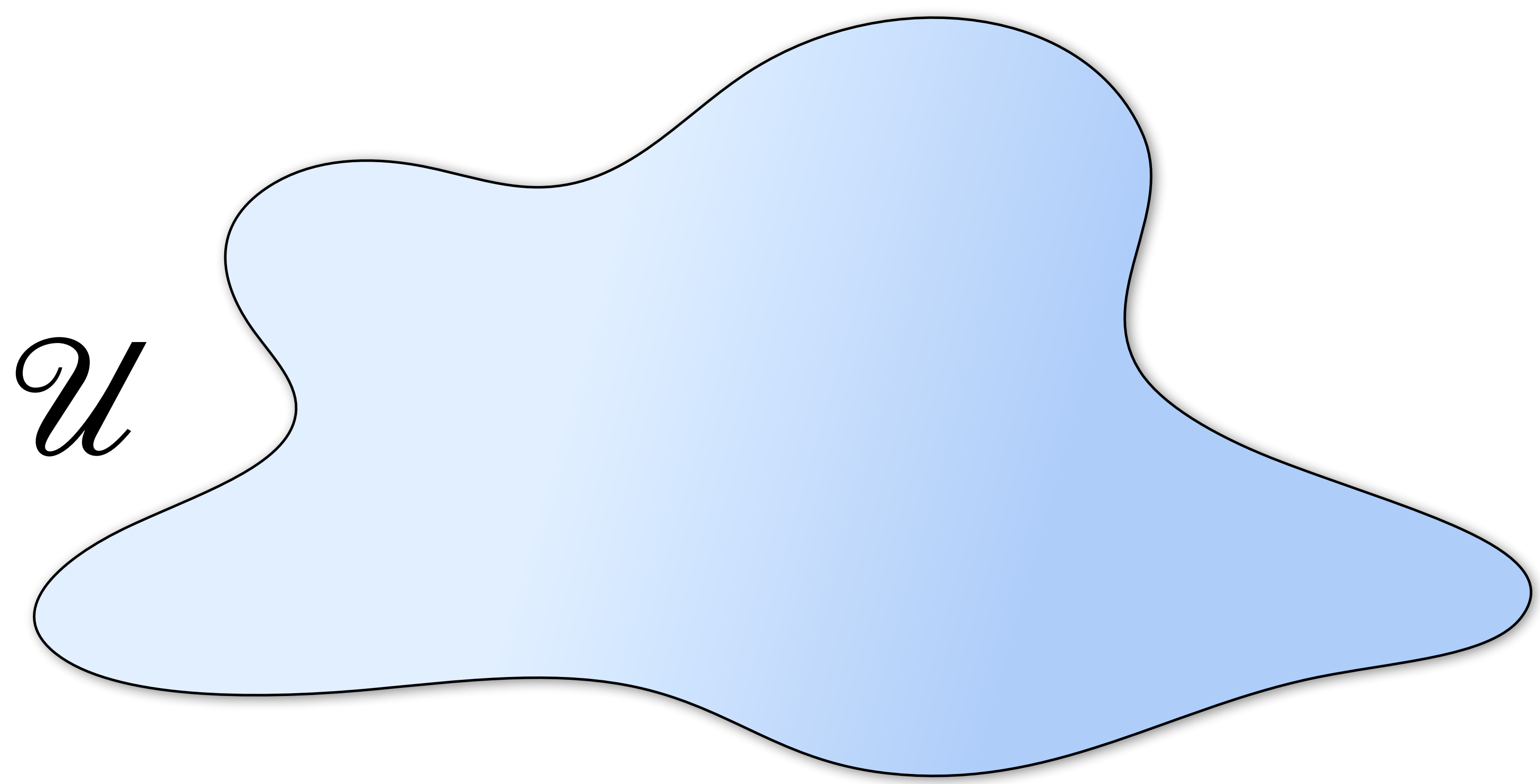
- A relatively simple case: the infinite-dimensional algebra $\mathfrak{A}$ is a limit of matrix algebras



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- A relatively simple case: the infinite-dimensional algebra $\mathfrak{A}$ is a limit of matrix algebras
- One may think the degrees of freedom in region $\mathscr{U}$ as an infinite collection of qubits.



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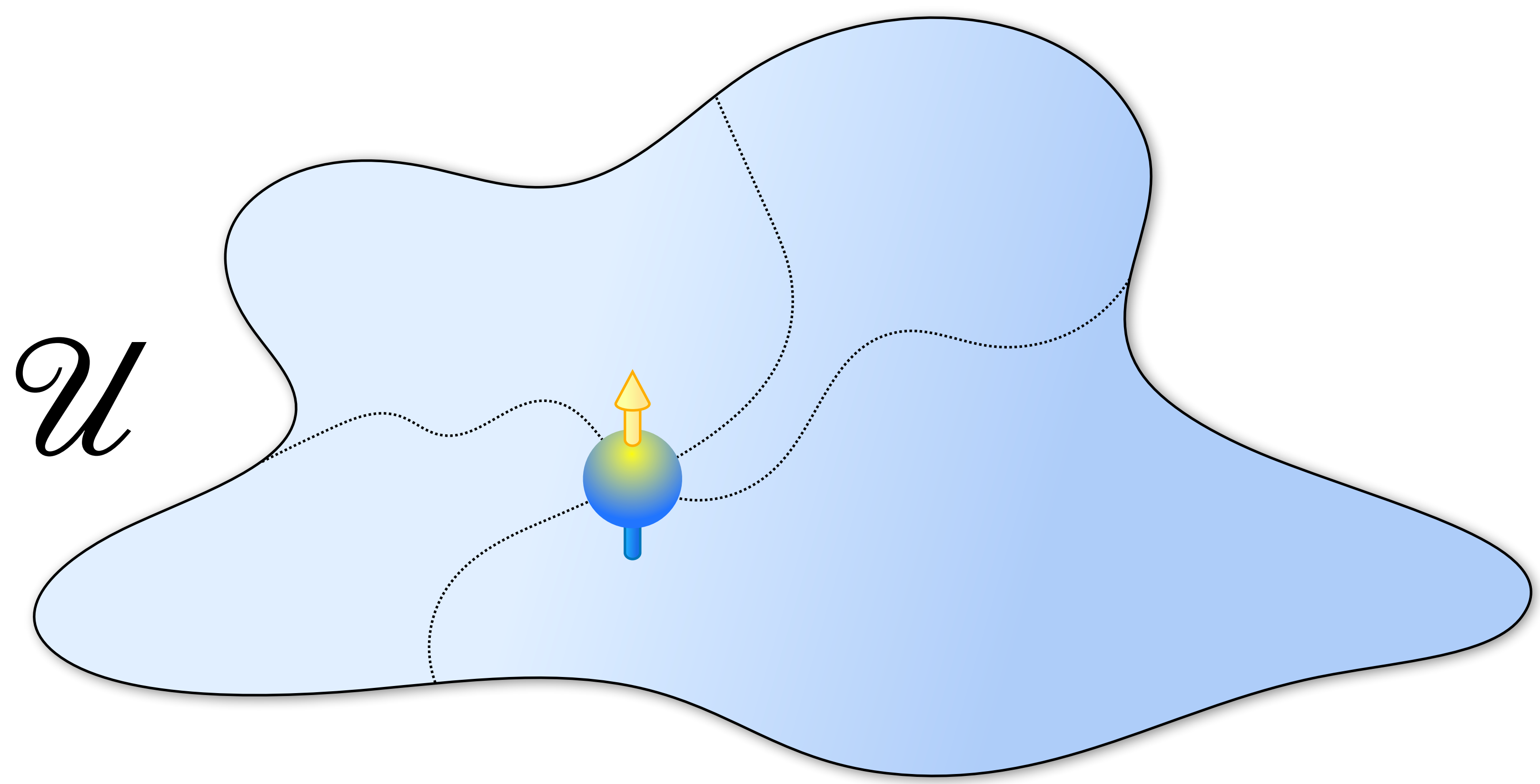

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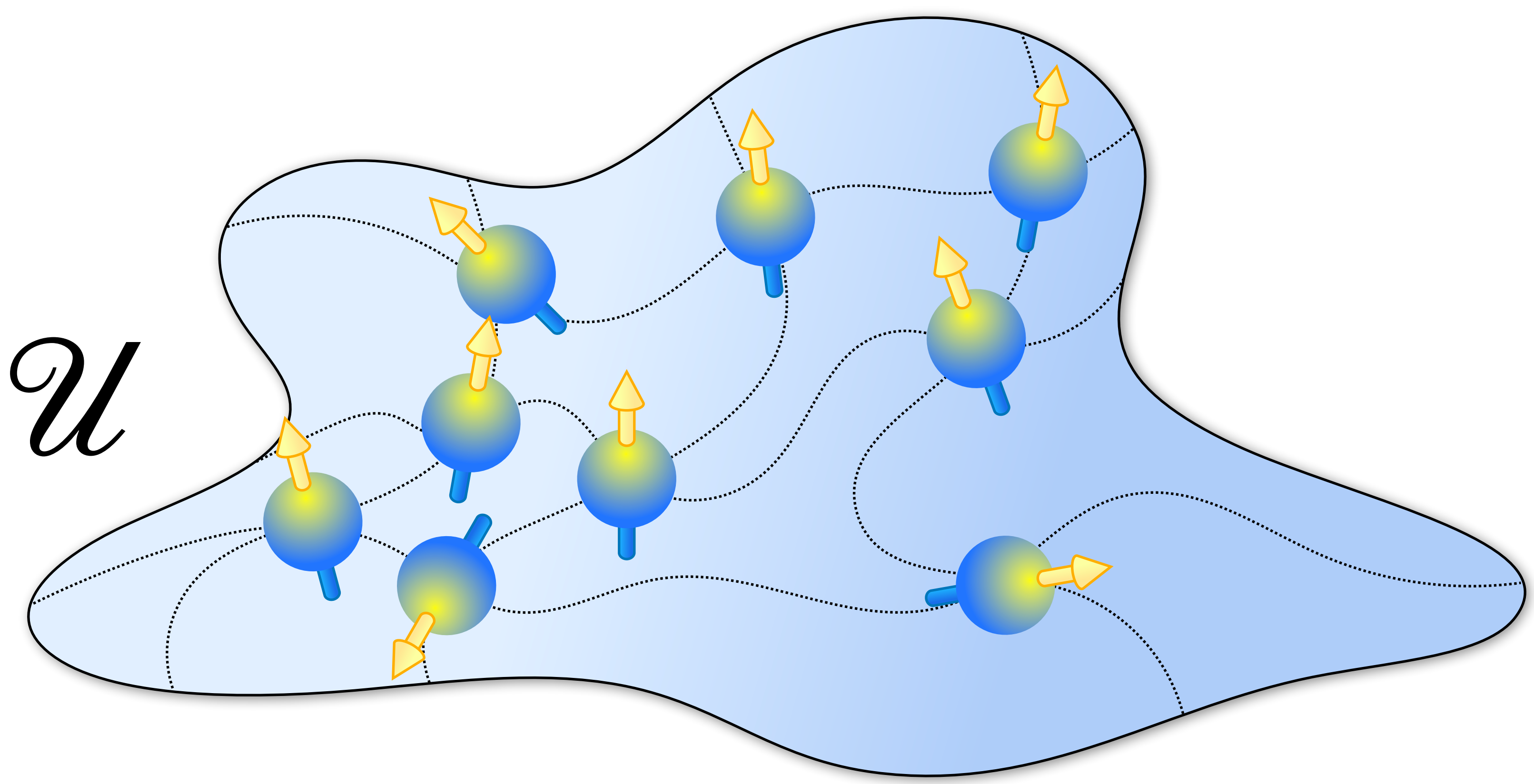

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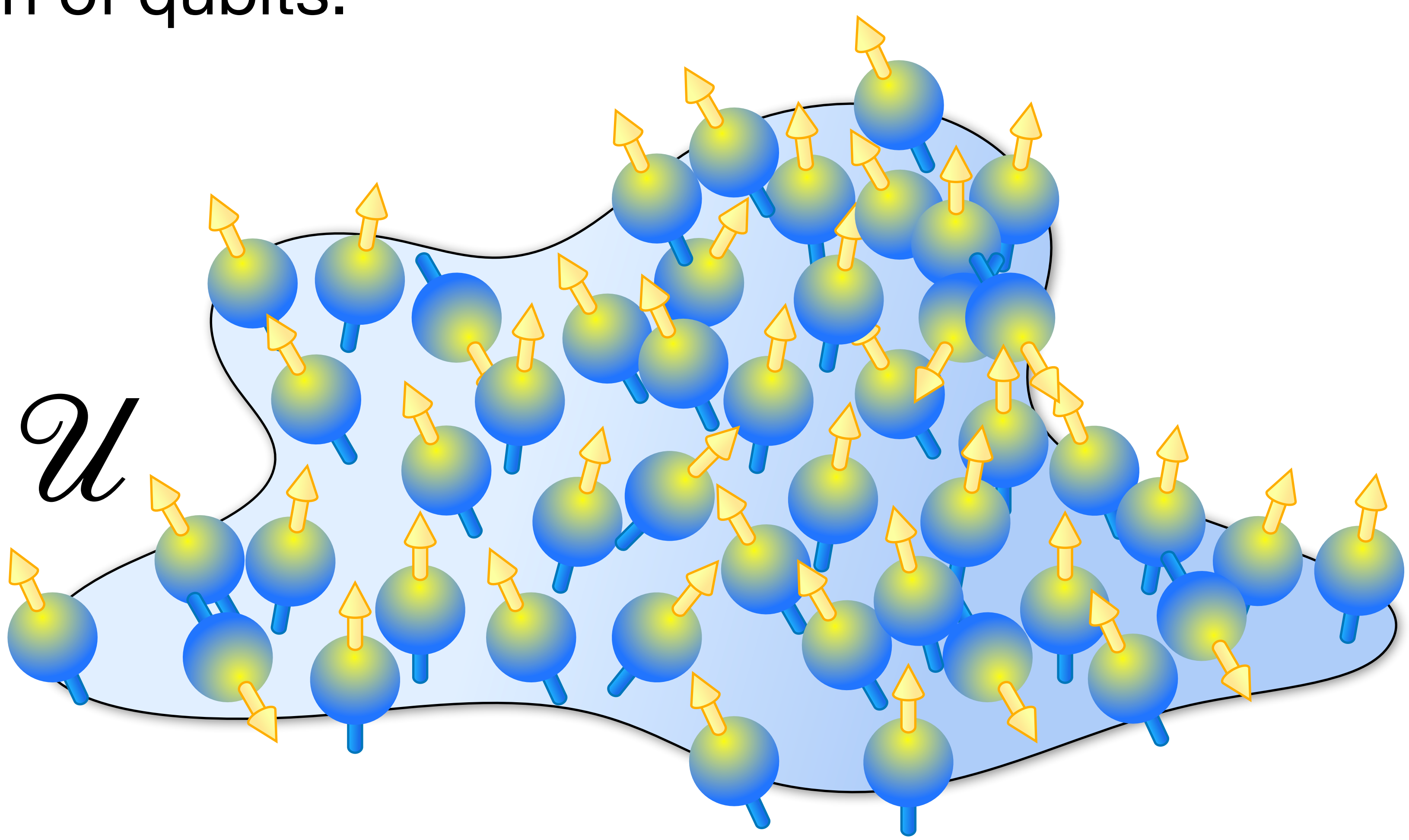

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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- A relatively simple case: the infinite-dimensional algebra $\mathfrak{A}$ is a limit of matrix algebras

- One may think the degrees of freedom in region $\mathscr{U}$ as an infinite collection of qubits. (Longo, 1978)

$$\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{M}_n \subset \cdots \subset \mathfrak{A}(\mathscr{U})$$

- This is believed that this picture is rigorously valid in quantum field theory.

- At each finite step in this chain, one defines an approximation $\Delta_\Psi^{(n)}$ to the modular operator (or similarly to J_Ψ or $\Delta_{\Psi|\Phi}$)

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The domain of Δ_Ψ^{is} to the modular operator (or $\Delta_{\Psi|\Phi}^{is}$):
  - For a matrix algebra, $\Delta_\Psi^{iz} = \exp(iz \log \Delta_\Psi)$ is an **entire matrix-valued function** of z;

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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- The domain of Δ_Ψ^{is} (or $\Delta_{\Psi|\Phi}^{is}$):
  - For a matrix algebra, $\Delta_\Psi^{iz} = \exp(iz \log \Delta_\Psi)$ is an **entire matrix-valued function** of z;
  - In quantum field theory, Δ_Ψ is unbounded and the analytic properties of $\Delta_\Psi^{iz}|\psi\rangle$ for a state $|\psi\rangle$ depend very much on $|\psi\rangle$:
    - One can find $|\psi\rangle$ such that $\Delta_\Psi^{iz}|\psi\rangle$ in entire in z;
    - One may also find some extreme $|\psi\rangle$ on which $\Delta_\Psi^{iz}|\psi\rangle$ can only be defined for real z.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The domain of Δ_{Ψ}^{is} (or $\Delta_{\Psi|\Phi}^{is}$)

- How about the domain when Δ_{Ψ}^{is} acts on $\mathbf{a}|\Psi\rangle$ ($\mathbf{a} \in \mathfrak{A}$ or $\mathbf{a}'|\Psi\rangle$, $\mathbf{a}' \in \mathfrak{A}'$)?

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$$\left|\Delta_{\Psi}^{1/2}\mathbf{a}\,|\,\Psi\rangle\right|^2 = \langle \Delta_{\Psi}^{1/2}\mathbf{a}\Psi\,|\,\Delta_{\Psi}^{1/2}\mathbf{a}\Psi\rangle = \langle \mathbf{a}\Psi\,|\,\Delta_{\Psi}\,|\,\mathbf{a}\Psi\rangle$$

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$$= \langle \mathbf{a}\Psi\,|\,S_{\Psi}^{\dagger}S_{\Psi}\,|\,\mathbf{a}\Psi\rangle = \overline{\langle S_{\Psi}\mathbf{a}\Psi\,|\,S_{\Psi}\mathbf{a}\Psi\rangle}$$

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$$
\begin{aligned}
\left|\Delta_\Psi^{1/2}\mathbf{a}\,|\,\Psi\rangle\right|^2 &= \langle \Delta_\Psi^{1/2}\mathbf{a}\Psi\,|\,\Delta_\Psi^{1/2}\mathbf{a}\Psi\rangle = \langle \mathbf{a}\Psi\,|\,\Delta_\Psi\,|\,\mathbf{a}\Psi\rangle \\
&= \langle \mathbf{a}\Psi\,|\,S_\Psi^\dagger S_\Psi\,|\,\mathbf{a}\Psi\rangle = \overline{\langle S_\Psi\mathbf{a}\Psi\,|\,S_\Psi\mathbf{a}\Psi\rangle} \\
&= \overline{\langle \mathbf{a}^\dagger\Psi\,|\,\mathbf{a}^\dagger\Psi\rangle} < \infty
\end{aligned}
$$

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$$\left| \Delta_\Psi^{1/2}\mathbf{a}\,|\,\Psi\rangle \right|^2 = \overline{\langle \mathbf{a}^\dagger\Psi\,|\,\mathbf{a}^\dagger\Psi\rangle} < \infty$$

- Because $\lambda^r < \lambda + 1 \;\; (0 \leqslant r \leqslant 1)$ for a positive real number λ implies $\Delta_\Psi^r < \Delta_\Psi + 1$,

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$$\left|\Delta_{\Psi}^{1/2}\mathbf{a}\,|\,\Psi\rangle\right|^2 = \overline{\langle \mathbf{a}^{\dagger}\Psi\,|\,\mathbf{a}^{\dagger}\Psi\rangle} < \infty$$

- Because $\lambda^r < \lambda + 1 \;(0 \leqslant r \leqslant 1)$ for a positive real number λ implies $\Delta_{\Psi}^{r} < \Delta_{\Psi} + 1$,

$$\langle \Delta_{\Psi}^{r/2}\mathbf{a}\Psi\,|\,\Delta_{\Psi}^{r/2}\mathbf{a}\Psi\rangle < \langle \Delta_{\Psi}^{1/2}\mathbf{a}\Psi\,|\,\Delta_{\Psi}^{1/2}\mathbf{a}\Psi\rangle + \langle \mathbf{a}\Psi\,|\,\mathbf{a}\Psi\rangle < \infty$$

$$0 \leqslant r \leqslant 1$$

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- The domain of Δ_{Ψ}^{is} (or $\Delta_{\Psi|\Phi}^{is}$)

- How about the domain when Δ_{Ψ}^{is} acts on $\mathbf{a}|\Psi\rangle$ ($\mathbf{a} \in \mathfrak{A}$ or $\mathbf{a}'|\Psi\rangle$, $\mathbf{a}' \in \mathfrak{A}'$)?

- The unitary operator Δ_{Ψ}^{is} $(s \in \mathbb{R})$ does not change the norm of a state, so for $0 \leqslant r \leqslant 1/2$, $s \in \mathbb{R}$,

$$\left| \Delta_{\Psi}^{r+is} \mathbf{a} \Psi \right|^{2} < \infty$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The domain of Δ_Ψ^{is} (or $\Delta_{\Psi|\Phi}^{is}$)

- How about the domain when Δ_Ψ^{is} acts on $\mathbf{a}\,|\,\Psi\rangle$ ($\mathbf{a} \in \mathfrak{A}$ or $\mathbf{a}'\,|\,\Psi\rangle$, $\mathbf{a}' \in \mathfrak{A}'$)?

- The unitary operator Δ_Ψ^{is} $(s \in \mathbb{R})$ does not change the norm of a state, so for $0 \leqslant r \leqslant 1/2,\ s \in \mathbb{R}$, In Witten's paper, there is a typo below equation (4.41).

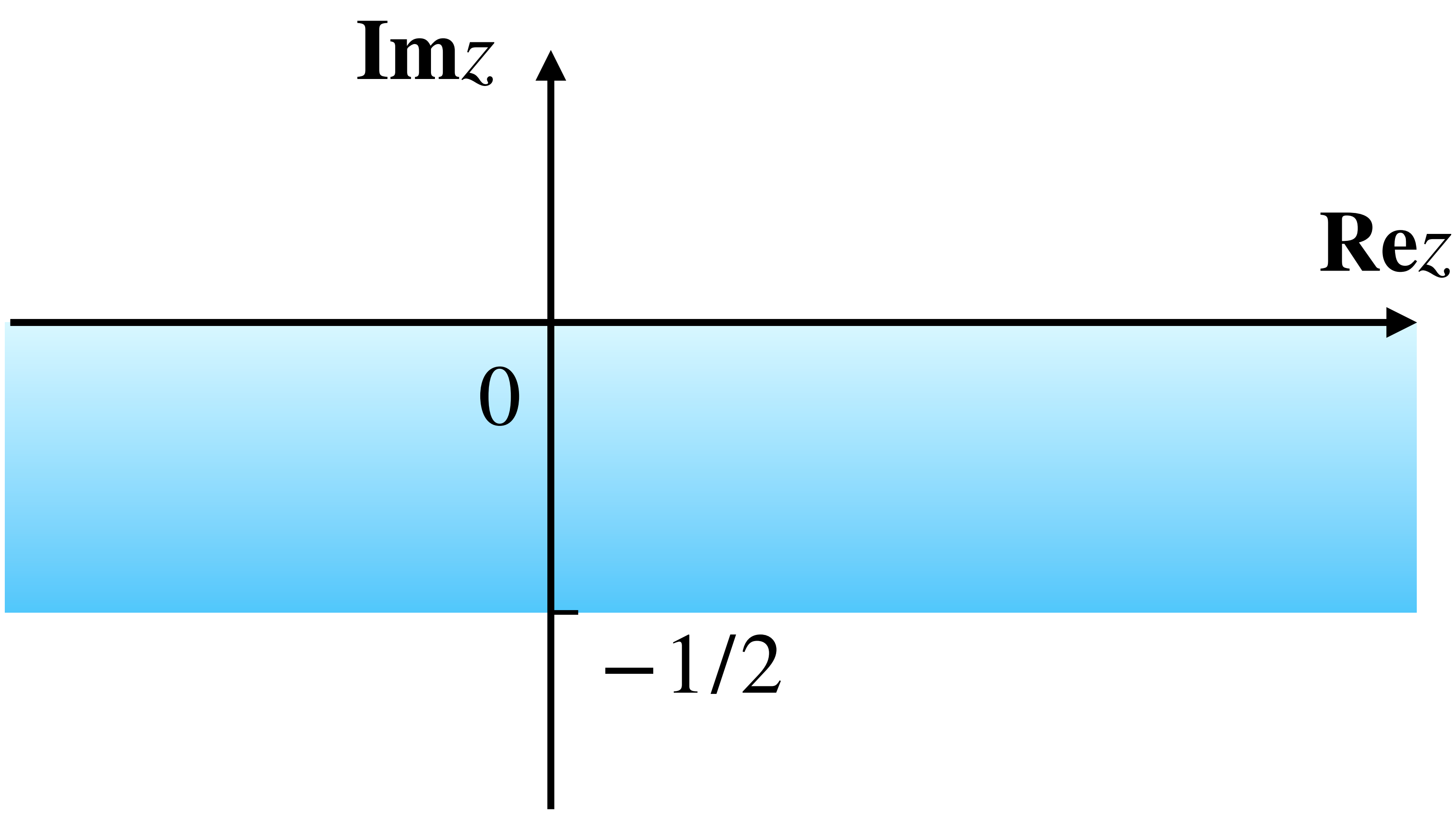
$$\left| \Delta_\Psi^{r+is} \mathbf{a} \Psi \right|^2 < \infty$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The domain of Δ_Ψ^{is} (or $\Delta_{\Psi|\Phi}^{is}$)

- $\Delta_\Psi^{iz}\mathbf{a}\,|\,\Psi\rangle$ is continuous in the strip $0 \geqslant \mathbf{Im}z \geqslant -1/2$ and holomorphic in the interior of the strip.



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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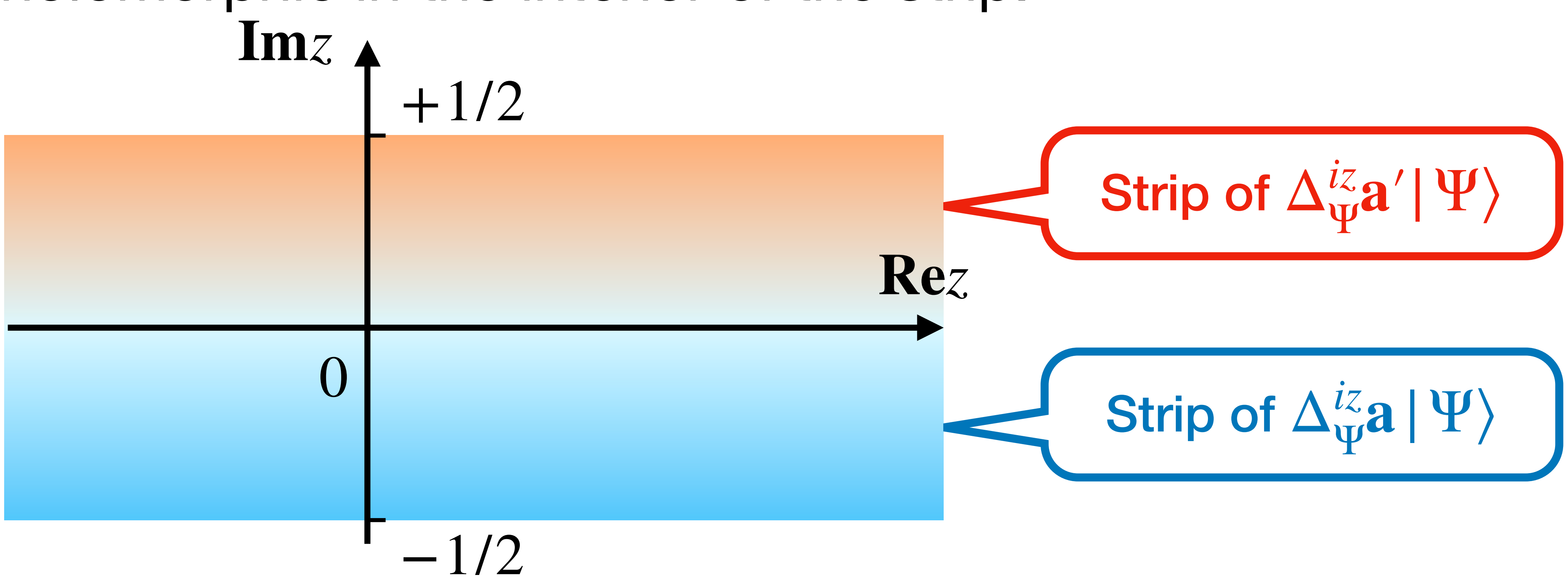

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The domain of Δ_{Ψ}^{is} (or $\Delta_{\Psi|\Phi}^{is}$)

- $\Delta_{\Psi}^{iz}\mathbf{a}\,|\,\Psi\rangle$ and $\Delta_{\Psi}^{iz}\mathbf{a}'\,|\,\Psi\rangle$ cannot be continued outside the strips.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The analytic properties of $F(z) = \langle \Psi \,|\, \mathbf{b}\Delta_{\Psi}^{iz}\mathbf{a} \,|\, \Psi \rangle$
- Why should we be interested in these functions?

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The analytic properties of $F(z) = \langle \Psi \,|\, \mathbf{b} \Delta_\Psi^{iz} \mathbf{a} \,|\, \Psi \rangle$

- Why should we be interested in these functions?

- They are “two-point correlation functions” on the cyclic separating state $|\,\Psi\rangle$ with Δ_Ψ^{iz} insertion.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The analytic properties of $F(z) = \langle \Psi | \mathbf{b} \Delta_\Psi^{iz} \mathbf{a} | \Psi \rangle$
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- For $z = s - ir \ (s, r \in \mathbb{R})$,

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$$= \langle \Delta_\Psi^{r/2} \mathbf{b}^\dagger \Psi | \Delta_\Psi^{is} | \Delta_\Psi^{r/2} \mathbf{a} \Psi \rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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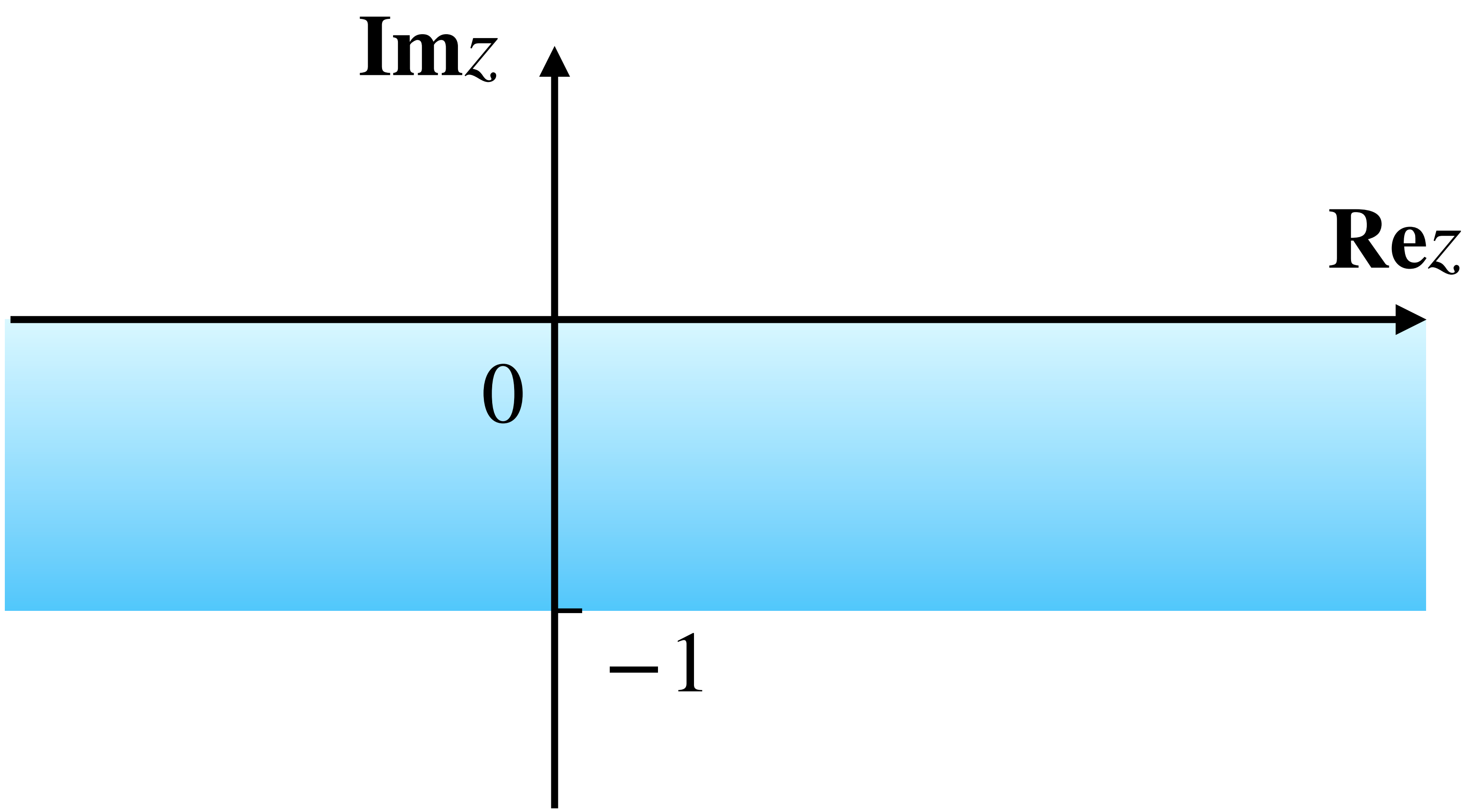

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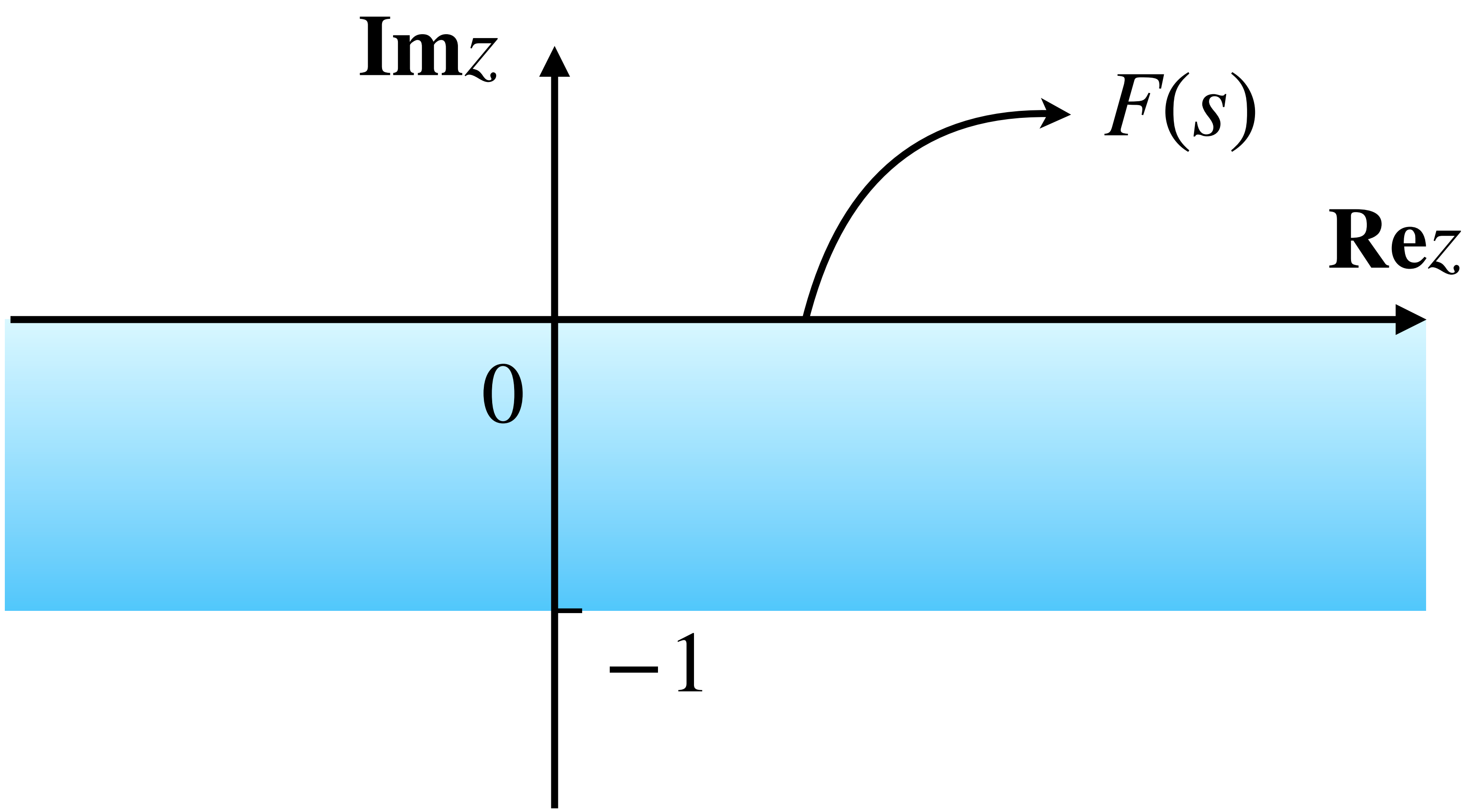

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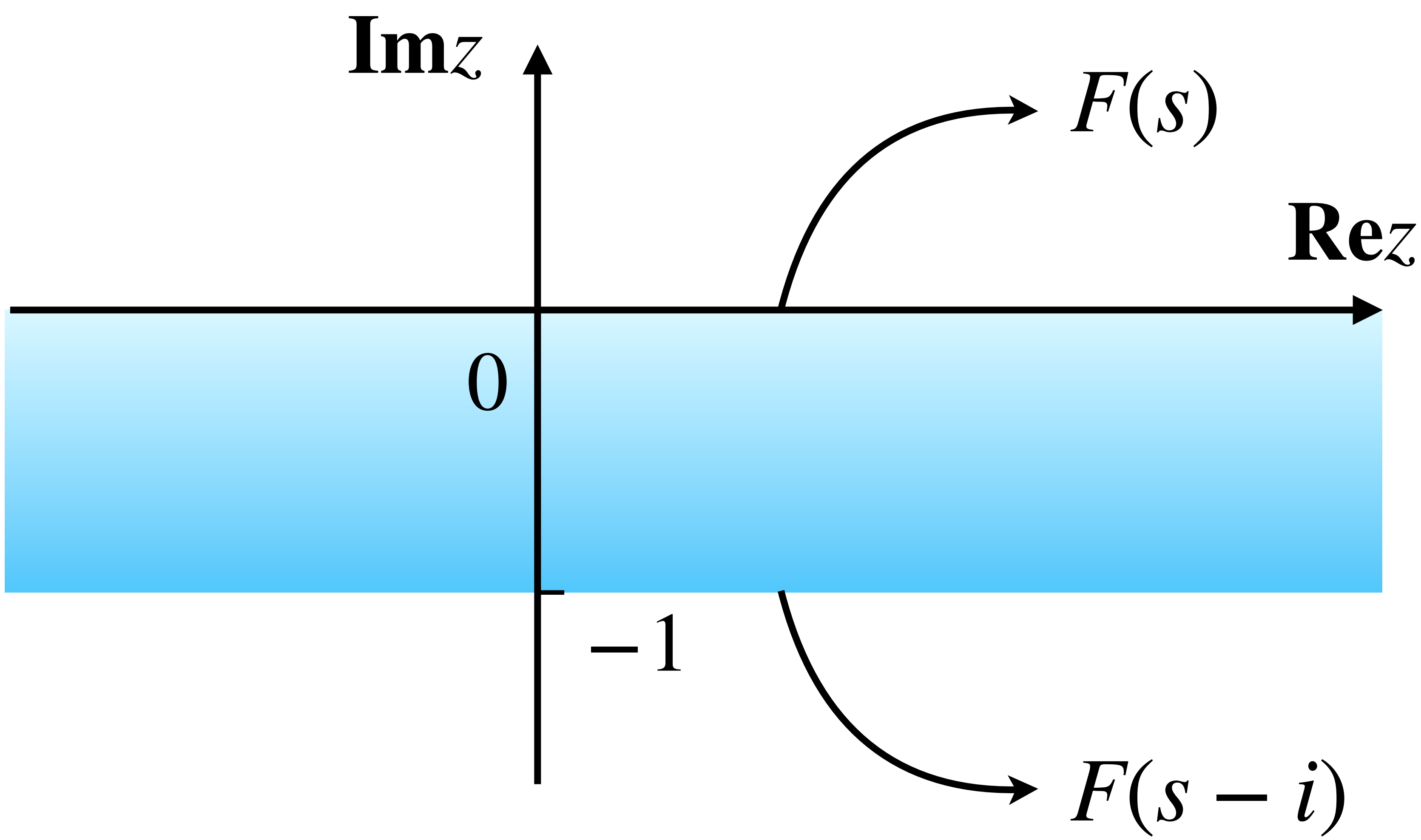

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$$= \langle J_\Psi S_\Psi \mathbf{b}^\dagger \Psi \,|\, \Delta_\Psi^{is} \,|\, J_\Psi S_\Psi \mathbf{a}\Psi \rangle = \langle J_\Psi \mathbf{b}\Psi \,|\, \Delta_\Psi^{is} \,|\, J_\Psi \mathbf{a}^\dagger \Psi \rangle$$

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&= \langle J_\Psi S_\Psi \mathbf{b}^\dagger \Psi \,|\, \Delta_\Psi^{is} \,|\, J_\Psi S_\Psi \mathbf{a}\Psi \rangle = \langle J_\Psi \mathbf{b}\Psi \,|\, \Delta_\Psi^{is} \,|\, J_\Psi \mathbf{a}^\dagger \Psi \rangle \\
&= \langle J_\Psi \mathbf{b}\Psi \,|\, J_\Psi \Delta_\Psi^{is}\mathbf{a}^\dagger\Psi \rangle = \langle \Delta_\Psi^{is}\mathbf{a}^\dagger\Psi \,|\, \mathbf{b}\Psi \rangle
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&= \langle \Psi \,|\, \mathbf{a}\Delta_\Psi^{-is}\mathbf{b} \,|\, \Psi \rangle
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$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The meaning of the analytic properties of $F(z) = \langle \Psi \,|\, \mathbf{b}\Delta_{\Psi}^{iz}\mathbf{a} \,|\, \Psi \rangle$

- Consider the bipartite system again, the density matrix of the subsystem 1 is $\hat{\rho}_1 = \mathbf{Tr}_2\hat{\rho}_{12} = \mathbf{Tr}_2(\,|\,\Psi\rangle\langle\Psi\,|\,)$, the expected value of any observable $\mathbf{a} \in \mathfrak{A}_1$ can be written as $\mathbf{Tr}_1(\hat{\rho}_1\mathbf{a})$.

- By quantum statistic physics, we know that the density matrix $\hat{\rho}$ of a balance system with Hamiltonian $\hat{H}$ and temperature $T = 1/\beta$ should be

$$\hat{\rho} = Z^{-1}\exp(-\beta\hat{H})$$

- So one can define a “modular Hamiltonian” $\hat{H}$ by $\hat{\rho}_1 = \exp(-\hat{H})$.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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$$= \langle \Psi \,|\, \mathbf{b}e^{-iz\hat{H}}\mathbf{a}e^{iz\hat{H}} \,|\, \Psi \rangle = \mathbf{Tr}_1\left[\mathbf{Tr}_2\left(\hat{\rho}_{12}\mathbf{b}e^{-iz\hat{H}}\mathbf{a}e^{iz\hat{H}}\right)\right]$$

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$$\begin{aligned} F(z) &= \langle \Psi \,|\, \mathbf{b}\Delta_\Psi^{iz}\mathbf{a} \,|\, \Psi \rangle = \langle \Psi \,|\, \mathbf{b}\Delta_\Psi^{iz}\mathbf{a}\Delta_\Psi^{-iz} \,|\, \Psi \rangle = \langle \Psi \,|\, \mathbf{b}\hat{\rho}_1^{iz}\mathbf{a}\hat{\rho}_1^{-iz} \,|\, \Psi \rangle \\ &= \langle \Psi \,|\, \mathbf{b}e^{-iz\hat{H}}\mathbf{a}e^{iz\hat{H}} \,|\, \Psi \rangle = \mathbf{Tr}_1 \left[\mathbf{Tr}_2 \left(\hat{\rho}_{12}\mathbf{b}e^{-iz\hat{H}}\mathbf{a}e^{iz\hat{H}} \right) \right] \\ &= \mathbf{Tr}_1 \left(\hat{\rho}_1 \mathbf{b}e^{-iz\hat{H}}\mathbf{a}e^{iz\hat{H}} \right) = \mathbf{Tr}_1 \left(e^{-\hat{H}}\mathbf{b}e^{-iz\hat{H}}\mathbf{a}e^{iz\hat{H}} \right) \end{aligned}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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$$F(s) = \mathbf{Tr}_1 \left[e^{-\hat{H}}\mathbf{b}\left(e^{-is\hat{H}}\mathbf{a}e^{is\hat{H}} \right) \right] = \mathbf{Tr}_1 \left[e^{-\hat{H}}\mathbf{ba}(-s) \right]$$

$$F(s-i) = \mathbf{Tr}_1 \left[e^{-\hat{H}}\left(e^{-is\hat{H}}\mathbf{a}e^{is\hat{H}} \right)\mathbf{b} \right] = \mathbf{Tr}_1 \left[e^{-\hat{H}}\mathbf{a}(-s)\mathbf{b} \right]$$

- Because $\mathbf{a}(s) = e^{is\hat{H}}\mathbf{a}e^{-is\hat{H}}$ is a Heisenberg operator at time s, these functions are real time two-point functions in a thermal ensemble with Hamiltonian $\hat{H}$ (with inverse temperature 1) with different operator orderings.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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$$F(z) = \mathbf{Tr}_1 \left(e^{-\hat{H}} \mathbf{b} e^{-iz\hat{H}} \mathbf{a} e^{iz\hat{H}} \right) = \mathbf{Tr}_1 \left(e^{-(1-iz)\hat{H}} \mathbf{b} e^{-iz\hat{H}} \mathbf{a} \right)$$

- For infinite-dimensional system $\mathscr{H}$ which can be factorized as $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, because the modular Hamiltonian $\hat{H}$ is inevitably unbounded, the trace is well-defined iff both iz and $1 - iz$ have non-negative real part, which means $0 \geqslant \mathbf{Im} z \geqslant -1$.

- This is in consistent with our result (without assuming the factorization of the Hilbert space).

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- Multi-point correlation functions, for example

$$F(z_1, z_2) = \mathbf{Tr}_1 \left(e^{-\hat{H}} \mathbf{c} e^{-iz_1\hat{H}} \mathbf{b} e^{-i(z_2 - z_1)\hat{H}} \mathbf{a} e^{iz_2\hat{H}} \right)$$

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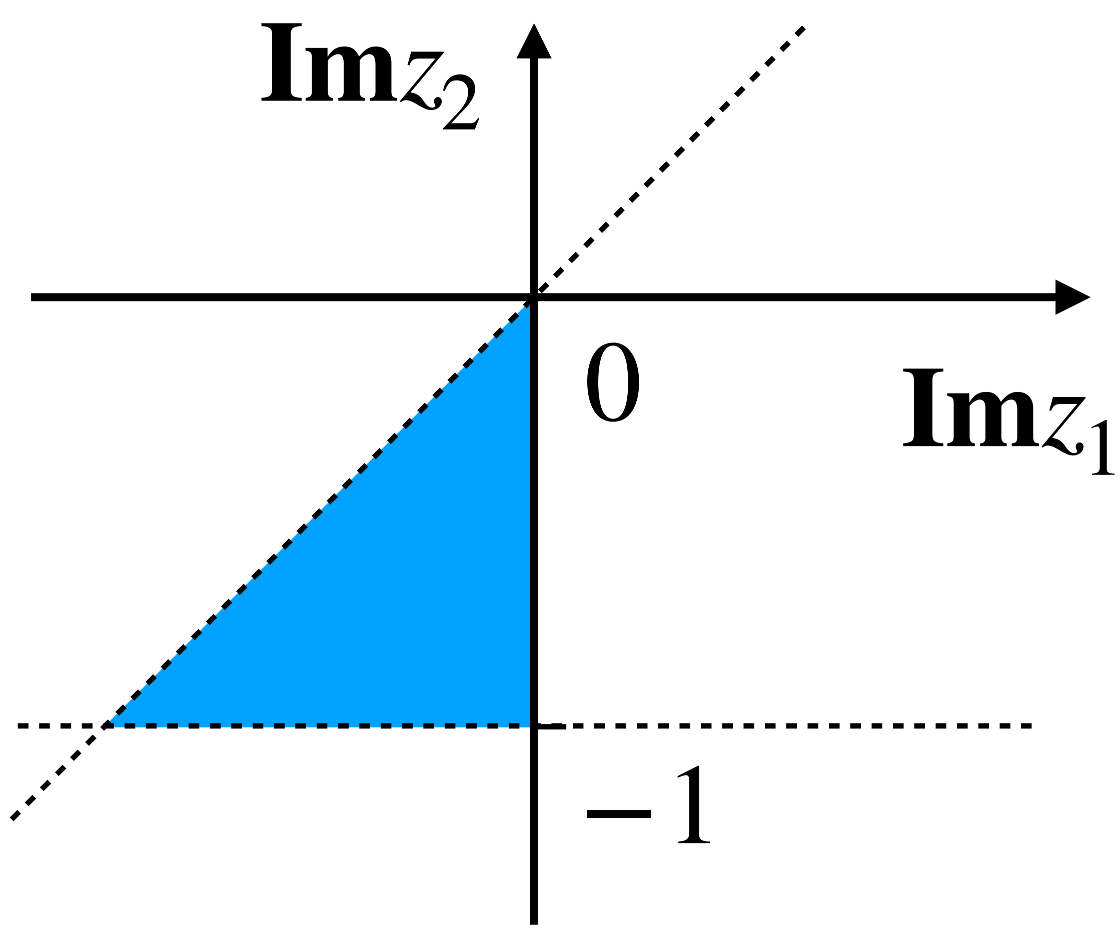

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FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

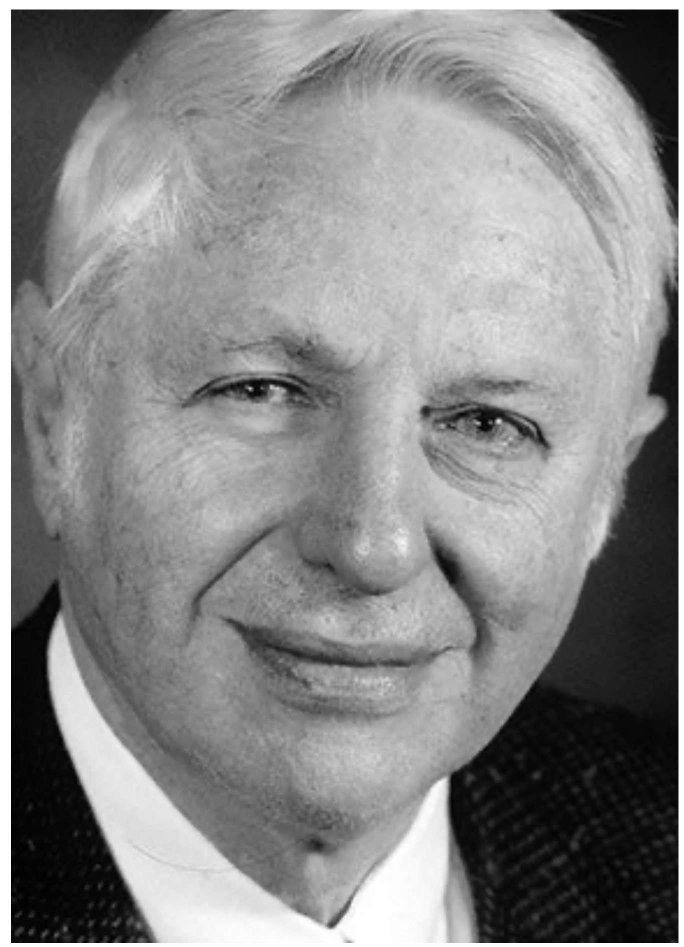
- All statements about holomorphy still apply if Δ_Ψ is replaced by the relative modular operator $\Delta_{\Psi|\Phi}$.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The KMS condition and KMS state ω (Kubo 1957, Martin and Schwinger 1959)

Ryogo Kubo
久保 亮五
(1920/02/15-1995/03/31)



Paul Cecil Martin
(1931/01/31-2016/06/19)

Julian Seymour
Schwinger
(1918/02/12-1994/07/16)

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

II. The modular automorphism group

- The KMS condition and KMS state ω (Kubo 1957, Martin and Schwinger 1959)

$$\mathbf{Tr}\left[e^{-\beta\hat{H}}\left(e^{it\hat{H}}\mathbf{A}e^{-it\hat{H}}\right)\mathbf{B}\right] = \mathbf{Tr}\left[e^{-\beta\hat{H}}\mathbf{B}\left(e^{i(t+i\beta)\hat{H}}\mathbf{A}e^{-i(t+i\beta)\hat{H}}\right)\right]$$

- The Tomita-Takesaki theory gives a Gibbs state which satisfies the KMS condition.

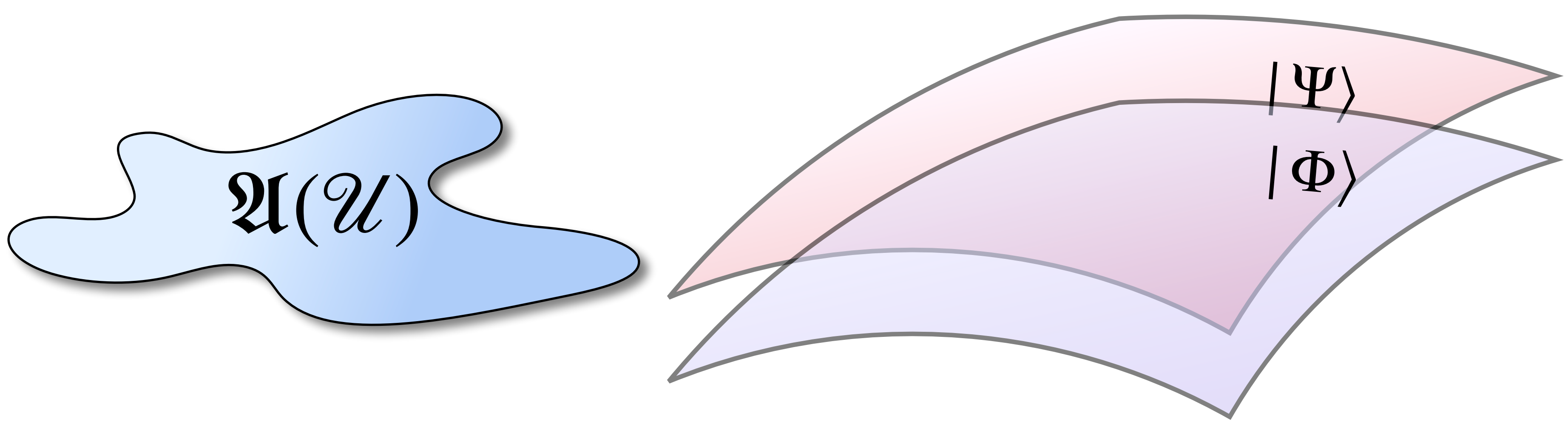
FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- Araki's definition of relative entropy: a spacetime region $\mathscr{U}$ and two states Ψ, Φ

$$\mathcal{S}_{\Psi|\Phi;\mathscr{U}} = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathscr{U}}|\Psi\rangle$$

- How does it go back to the usual definition of the relative entropy of a finite degrees of freedom system?



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- In nonrelativistic quantum mechanics, there is not spacetime region, but still commuting algebras $\mathfrak{A}$ and $\mathfrak{A}'$.

- Let Ψ be a cyclic separating vector for both $\mathfrak{A}$ and $\mathfrak{A}'$, and Φ be a second state vector. (The bipartite system again)

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$$\mathcal{S}_{\Psi|\Phi} = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle = -\mathbf{Tr}\Big(|\Psi\rangle\langle\Psi|\log\Delta_{\Psi|\Phi}\Big)$$

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$$\begin{aligned} \mathcal{S}_{\Psi|\Phi} &= -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle = -\mathbf{Tr}\left(|\Psi\rangle\langle\Psi|\log\Delta_{\Psi|\Phi}\right) \\ &= -\mathbf{Tr}_{12}\left(\rho_{12}\log\Delta_{\Psi|\Phi}\right) = -\mathbf{Tr}_{12}\left[\rho_{12}\log\left(\sigma_1\otimes\rho_2^{-1}\right)\right] \end{aligned}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- How to calculate $\log\left(\sigma_1 \otimes \rho_2^{-1}\right)$?

- To calculate the logarithm of a tensor product $\log(A \otimes B)$, we use singular value decomposition $A = U_A^\dagger \mathbf{diag}\{a_1, \cdots, a_n\} V_A$ and $B = U_B^\dagger \mathbf{diag}\{b_1, \cdots, b_n\} V_B$, then
$$\log(A \otimes B) = \log\left(U_A^\dagger \mathbf{diag}\{a_1, \cdots, a_n\} V_A \otimes U_B^\dagger \mathbf{diag}\{b_1, \cdots, b_n\} V_B\right)$$

- Under this base, the tensor product matrix is diagonalized to be $A \otimes B = \mathbf{diag}\{a_1 b_1, a_2 b_1, \cdots, a_n b_1, a_1 b_2, \cdots, a_n b_2, \cdots, \cdots, a_1 b_n, \cdots, a_n b_n\}$.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

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FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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$$\log(A \otimes B) = \mathbf{diag}\{\log a_1 + \log b_1, \log a_2 + \log b_1, \cdots, \log a_n + \log b_1, \log a_1 + \log b_2, \cdots,$$
$$\log a_n + \log b_2, \cdots, \cdots, \log a_1 + \log b_n, \cdots, \log a_n + \log b_n\}$$

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$$= \mathbf{diag}\{\log a_1, \log a_2, \log a_1, \log a_2, \cdots, \log a_n, \cdots, \log a_n, \cdots, \log a_1, \log a_2, \cdots, \log a_n\}$$
$$+\mathbf{diag}\{\log b_1, \cdots, \log b_1, \log b_2, \cdots, \log b_2, \cdots, \log b_n, \cdots, \log b_n\}$$

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$$= \log A \otimes \mathbf{1} + \mathbf{1} \otimes \log B$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- In nonrelativistic quantum mechanics, there is not spacetime region, but still commuting algebras $\mathfrak{A}$ and $\mathfrak{A}'$.

- Let Ψ be a cyclic separating vector for both $\mathfrak{A}$ and $\mathfrak{A}'$, and Φ be a second state vector. (The bipartite system again)

$$\mathcal{S}_{\Psi|\Phi} = -\,\mathbf{Tr}_{12}\left[\rho_{12}\log\left(\sigma_1 \otimes \rho_2^{-1}\right)\right]$$

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&= -\,\mathbf{Tr}_1\left(\rho_1\log\sigma_1\right) + \mathbf{Tr}_2\left(\rho_2\log\rho_2\right)
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&= -\mathbf{Tr}_1\left(\rho_1\log\sigma_1\right)+\mathbf{Tr}_1\left(\rho_1\log\rho_1\right) \\
&= \mathbf{Tr}\rho_1\left(\log\rho_1-\log\sigma_1\right)
\end{aligned}
$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- In nonrelativistic quantum mechanics, the relative entropy between two states with density matrices ρ_1 and σ_1 in Hilbert space $\mathscr{H}_1$ is

$$\mathcal{S}(\rho_1 \| \sigma_1) = \mathbf{Tr}\rho_1 \left(\log \rho_1 - \log \sigma_1\right)$$

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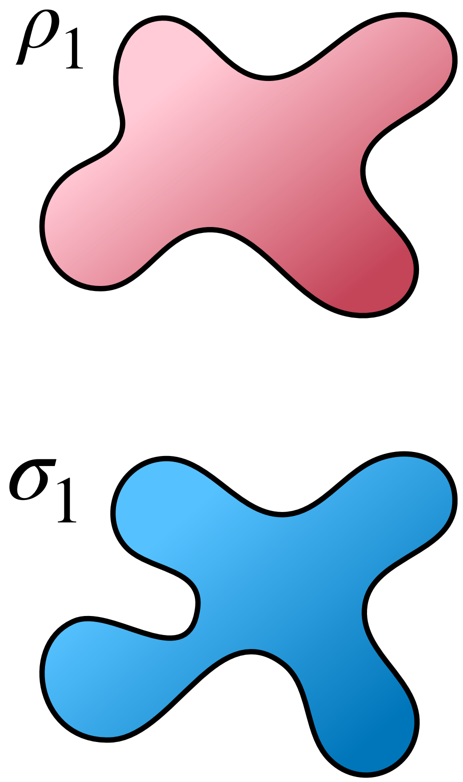
$$\mathcal{S}(\rho_1 \| \sigma_1) = \mathcal{S}_{\Psi|\Phi}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case



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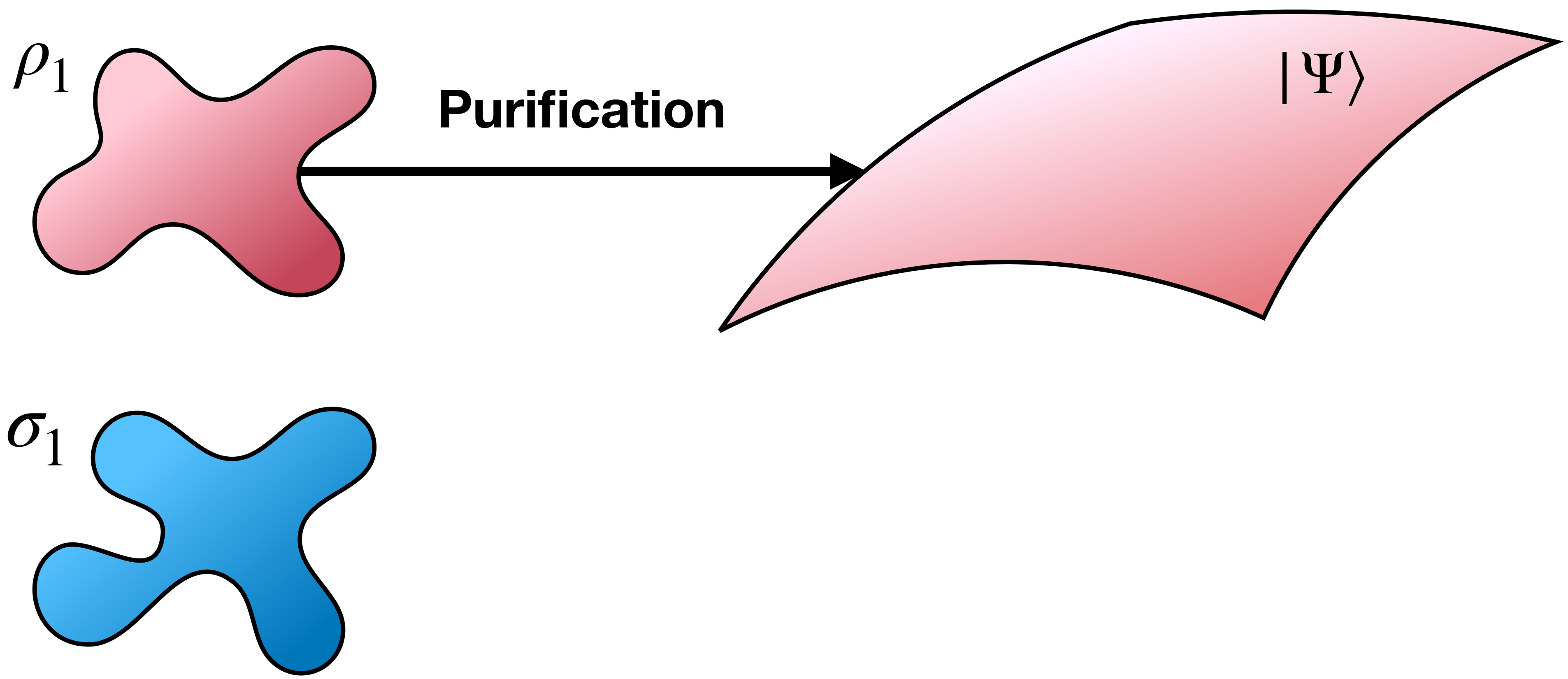

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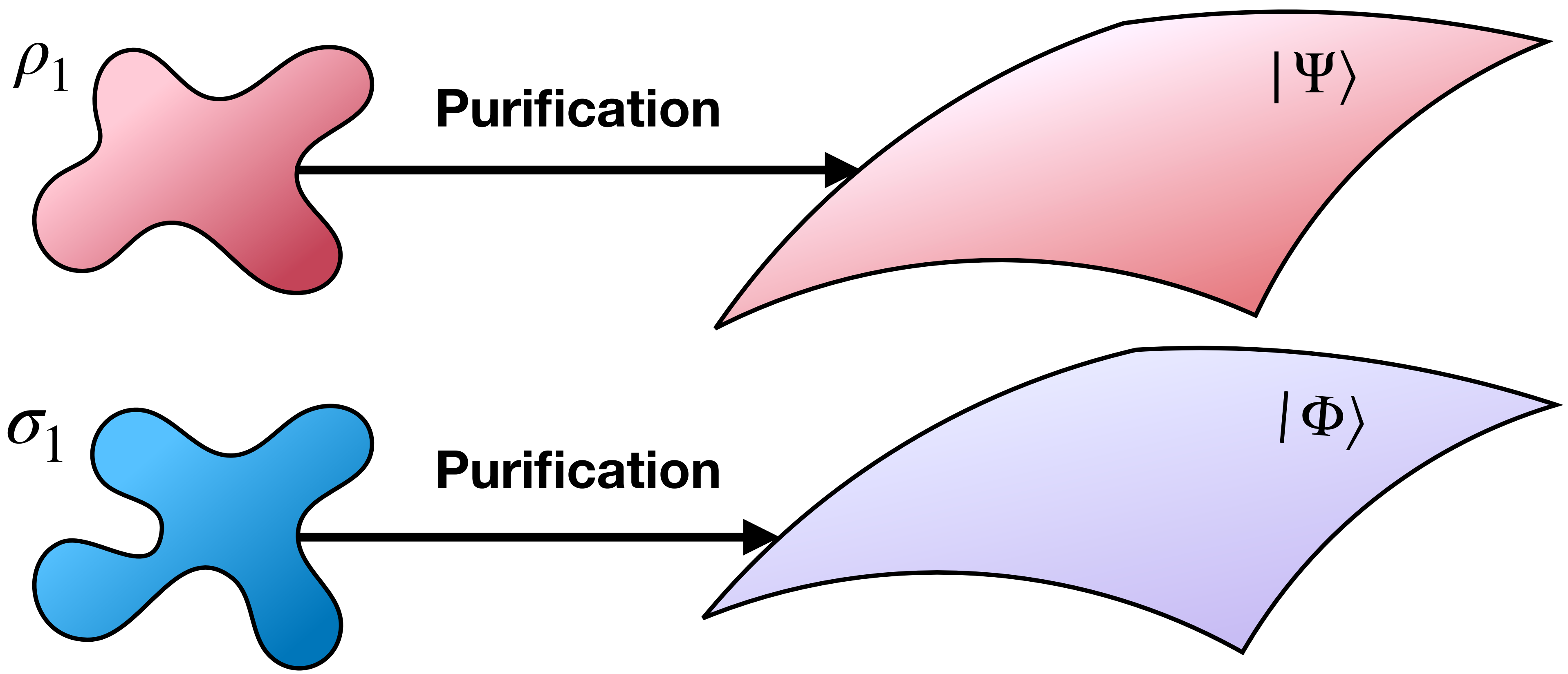

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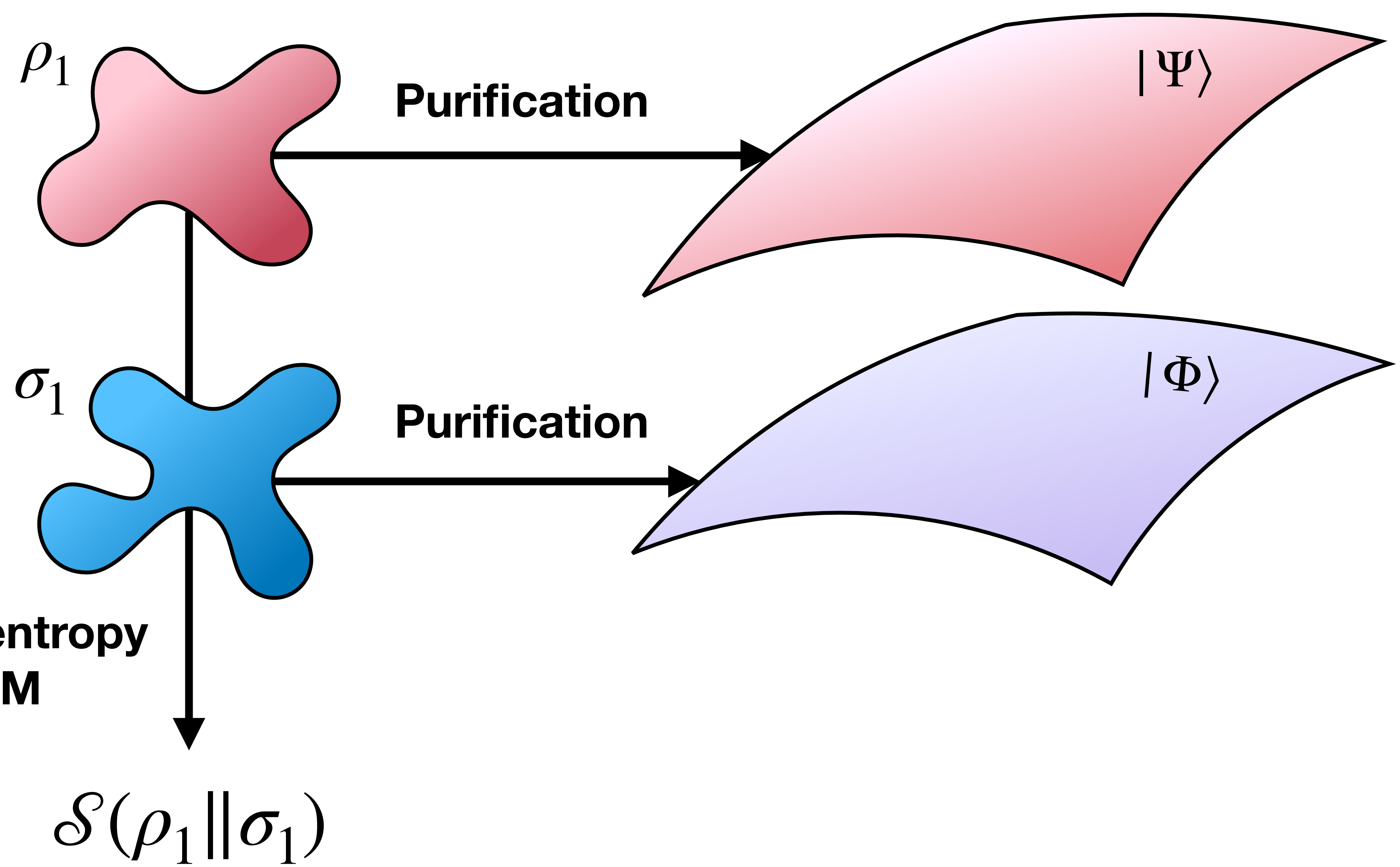

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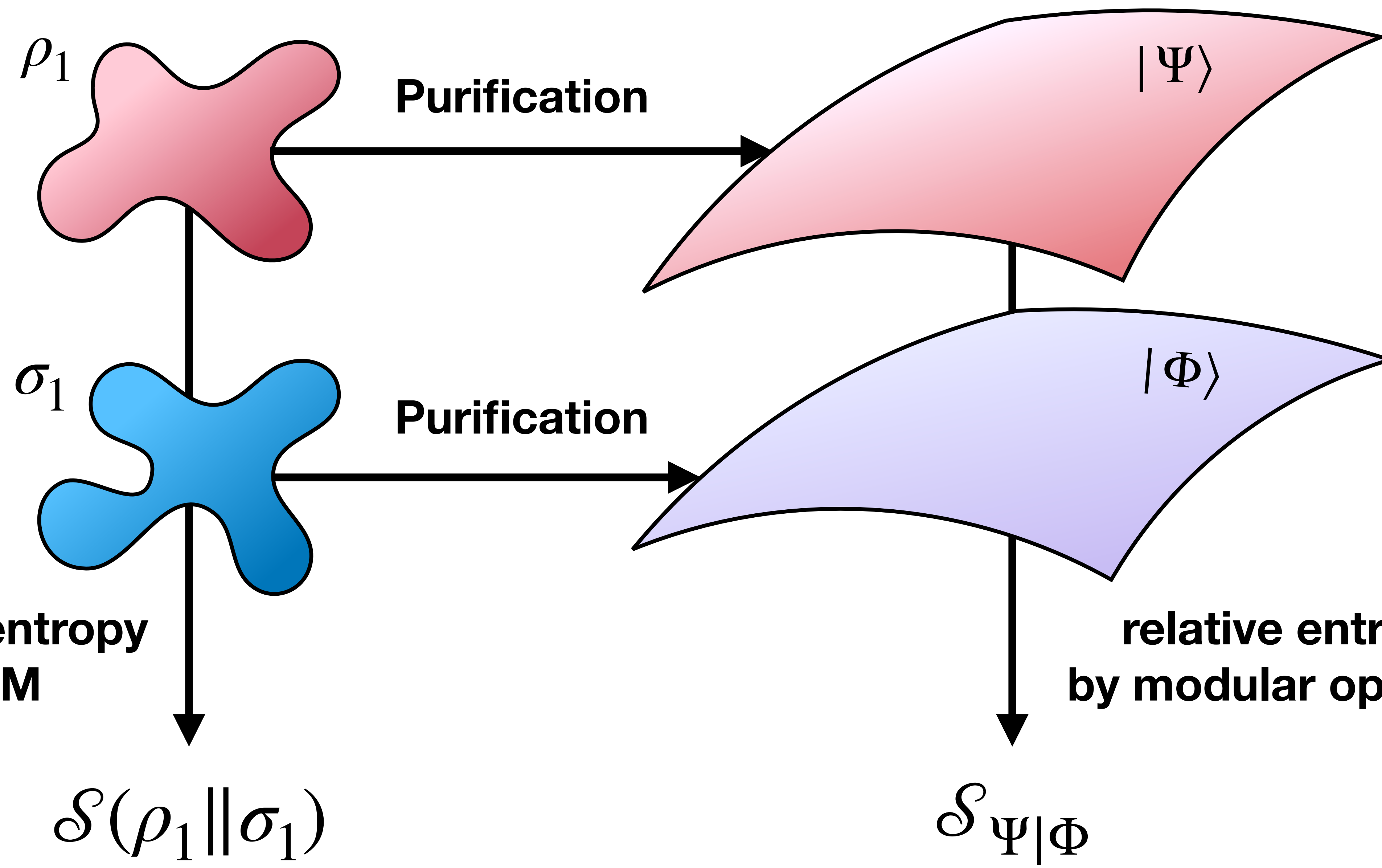

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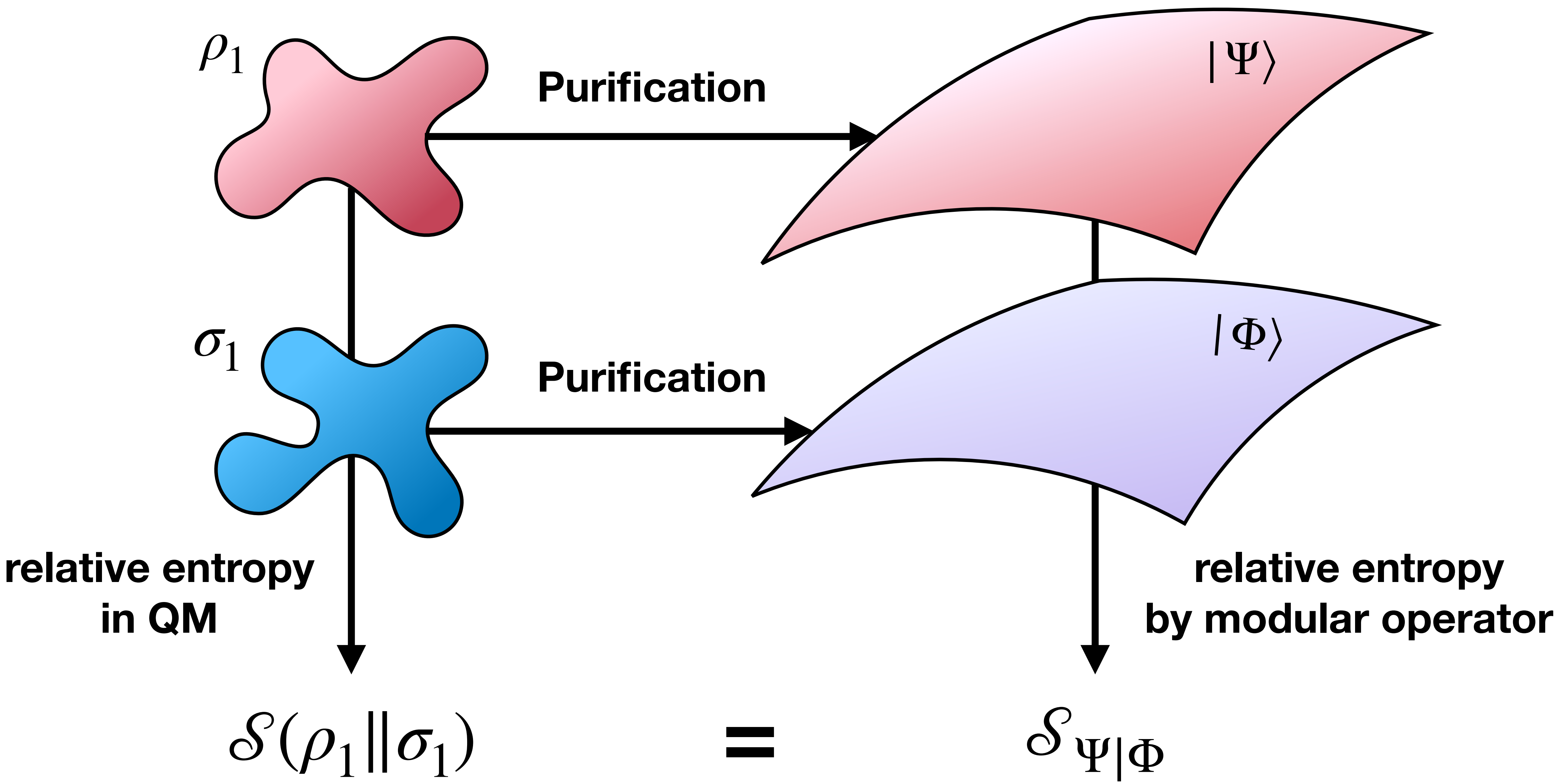


FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

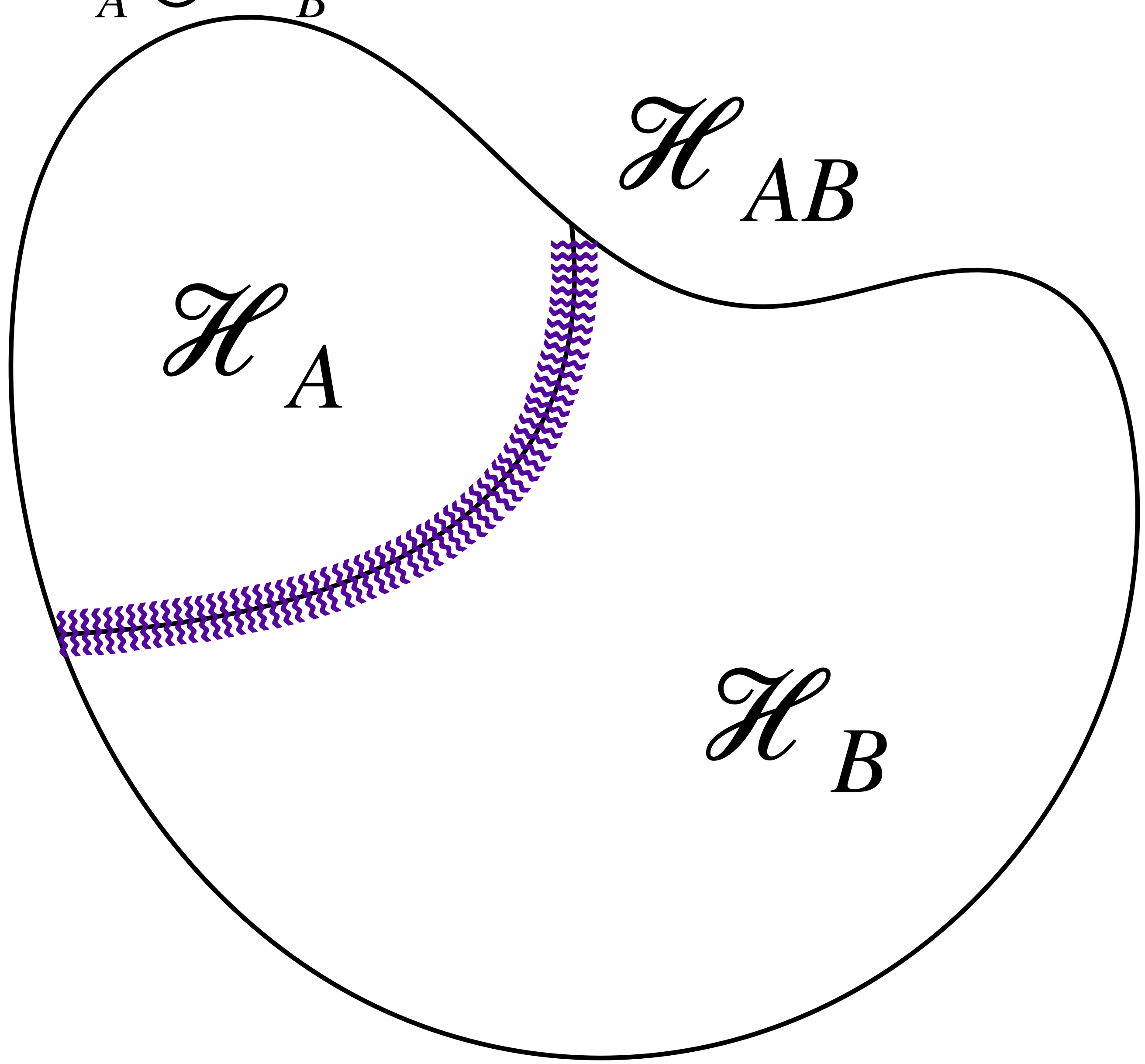
III. Monotonicity of relative entropy in the finite-dimensional case

- The important generic properties of relative entropy holds certainly in the (simple) nonrelativistic quantum mechanics case
  - Positivity;
  - monotonicity (?)
- How to understand the monotonicity in the nonrelativistic quantum mechanics case? (There is no spacetime region.)

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- In nonrelativistic quantum mechanics, one consider the Hilbert space $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- In nonrelativistic quantum mechanics, one consider the Hilbert space $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$

- Given density matrices ρ_{AB} and σ_{AB} on $\mathscr{H}_{AB}$, then one has reduced density matrices $\rho_A = \mathbf{Tr}_B \rho_{AB}$ and $\sigma_A = \mathbf{Tr}_B \sigma_{AB}$ on $\mathscr{H}_A$.

- The monotonicity of relative entropy is the relation between the relative entropies $\mathcal{S}(\rho_{AB} \| \sigma_{AB})$ and $\mathcal{S}(\rho_A \| \sigma_A)$,

$$\mathcal{S}(\rho_{AB} \| \sigma_{AB}) \geqslant \mathcal{S}(\rho_A \| \sigma_A)$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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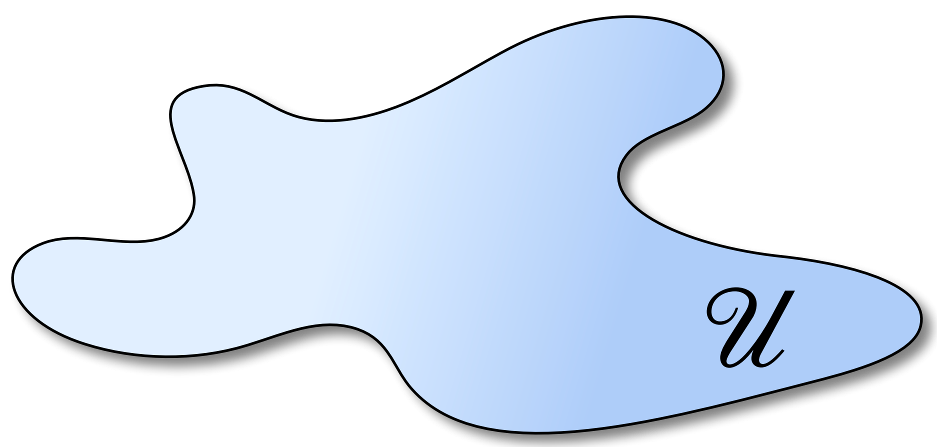

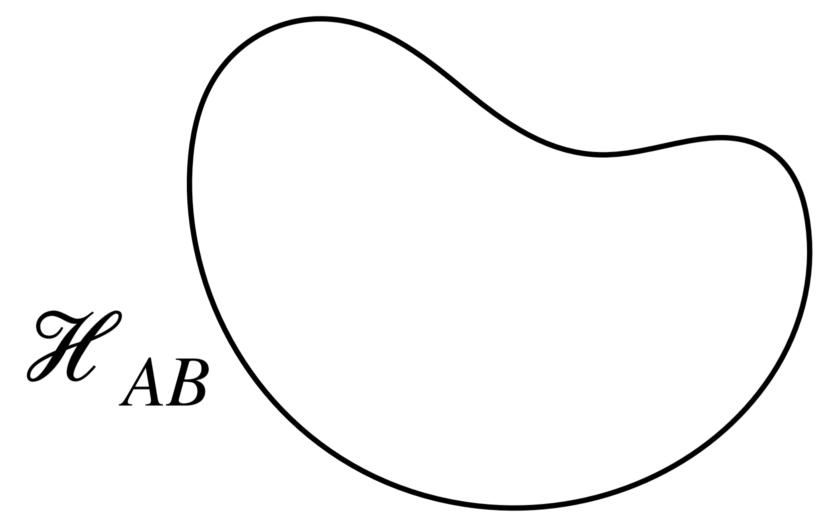

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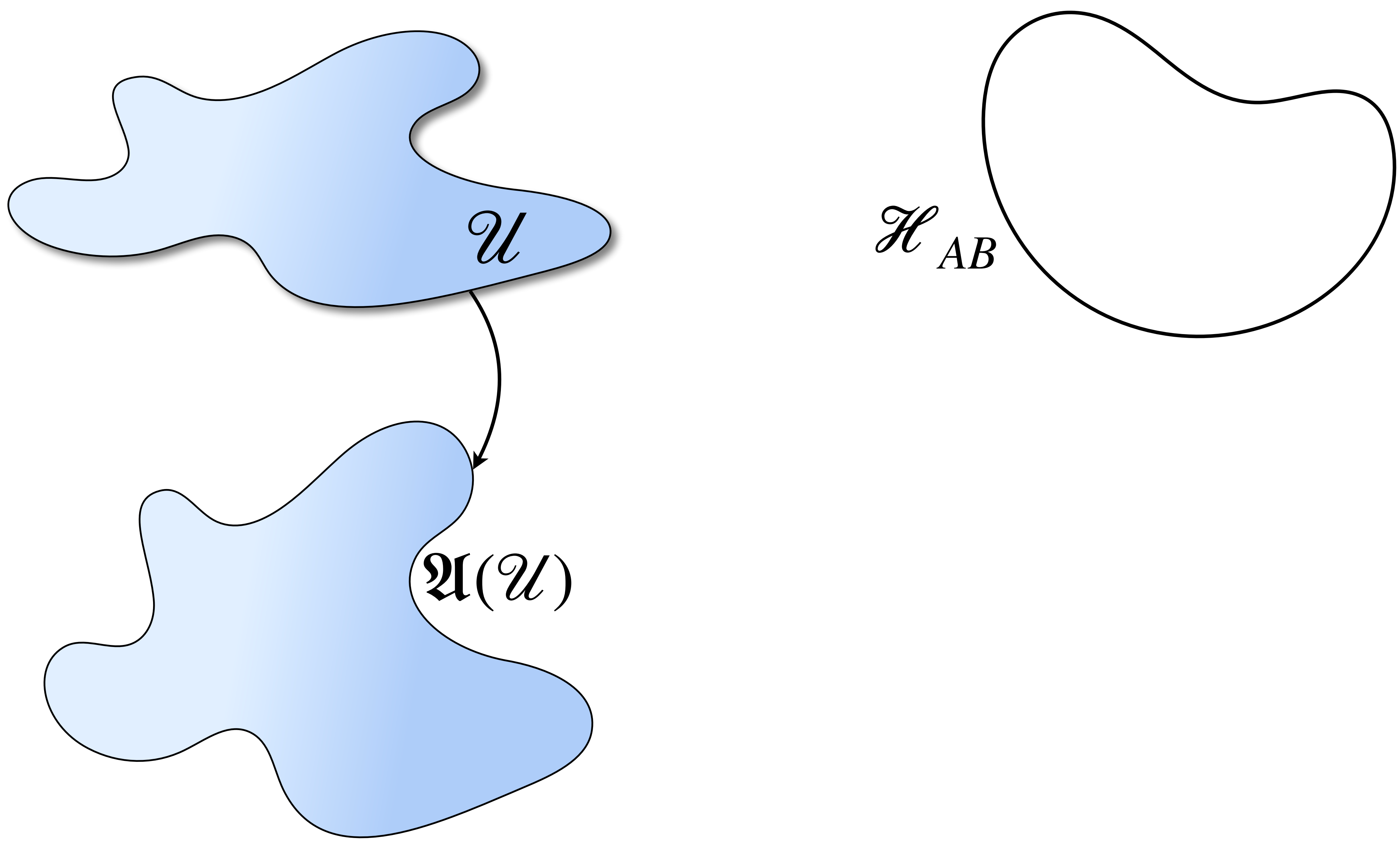

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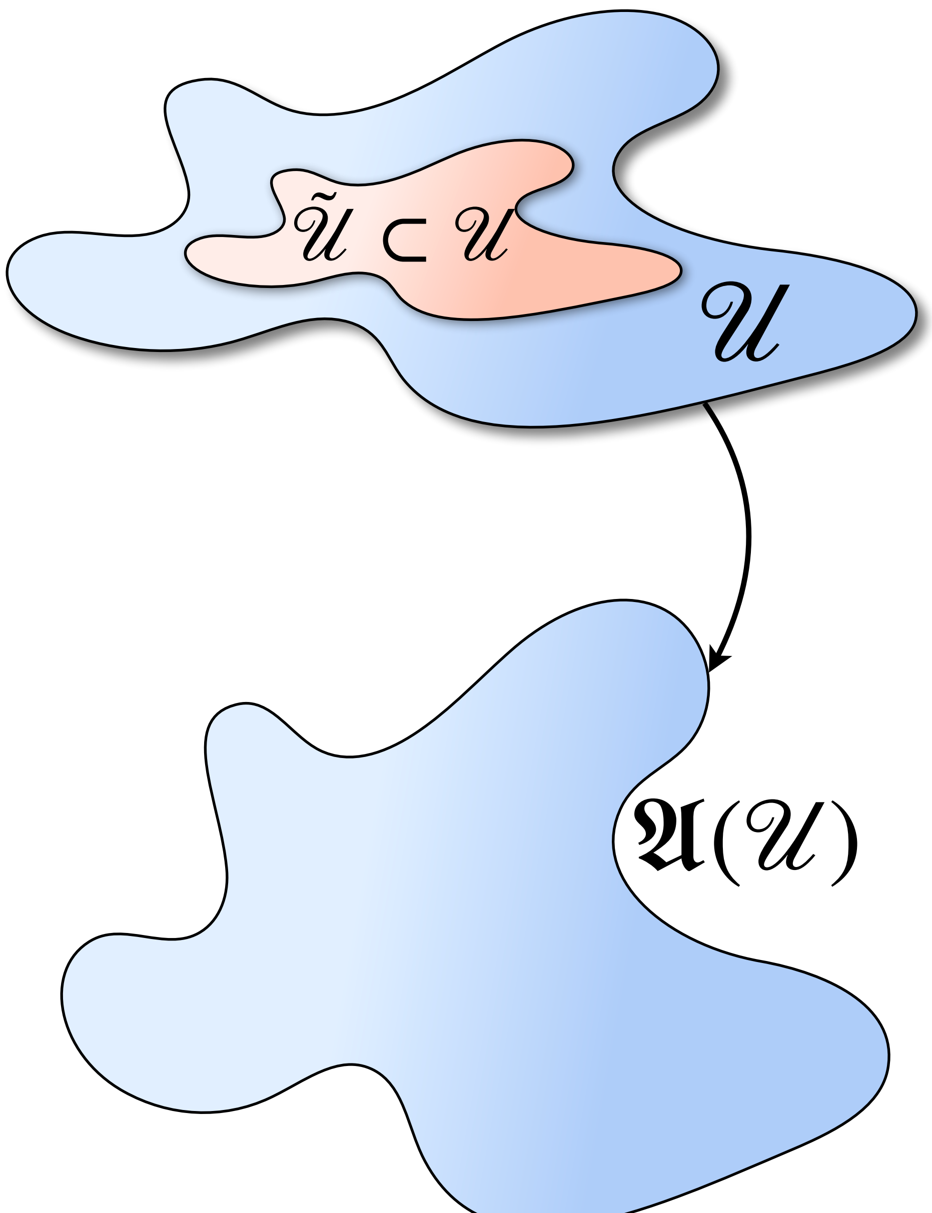

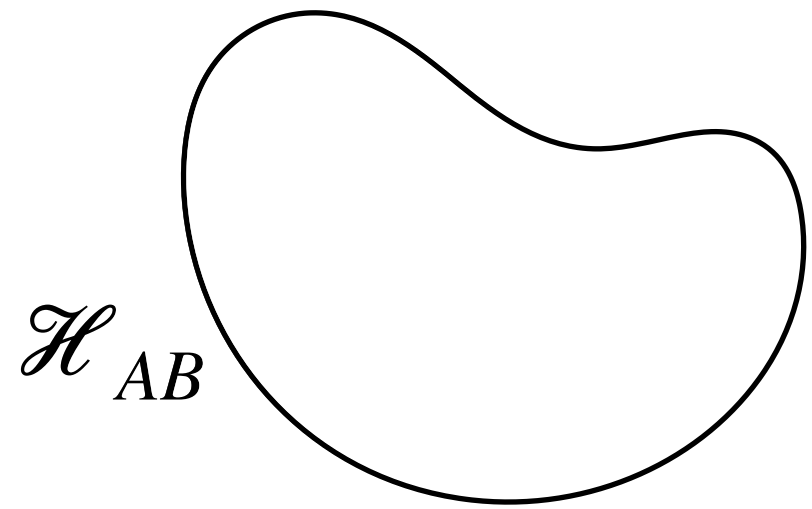

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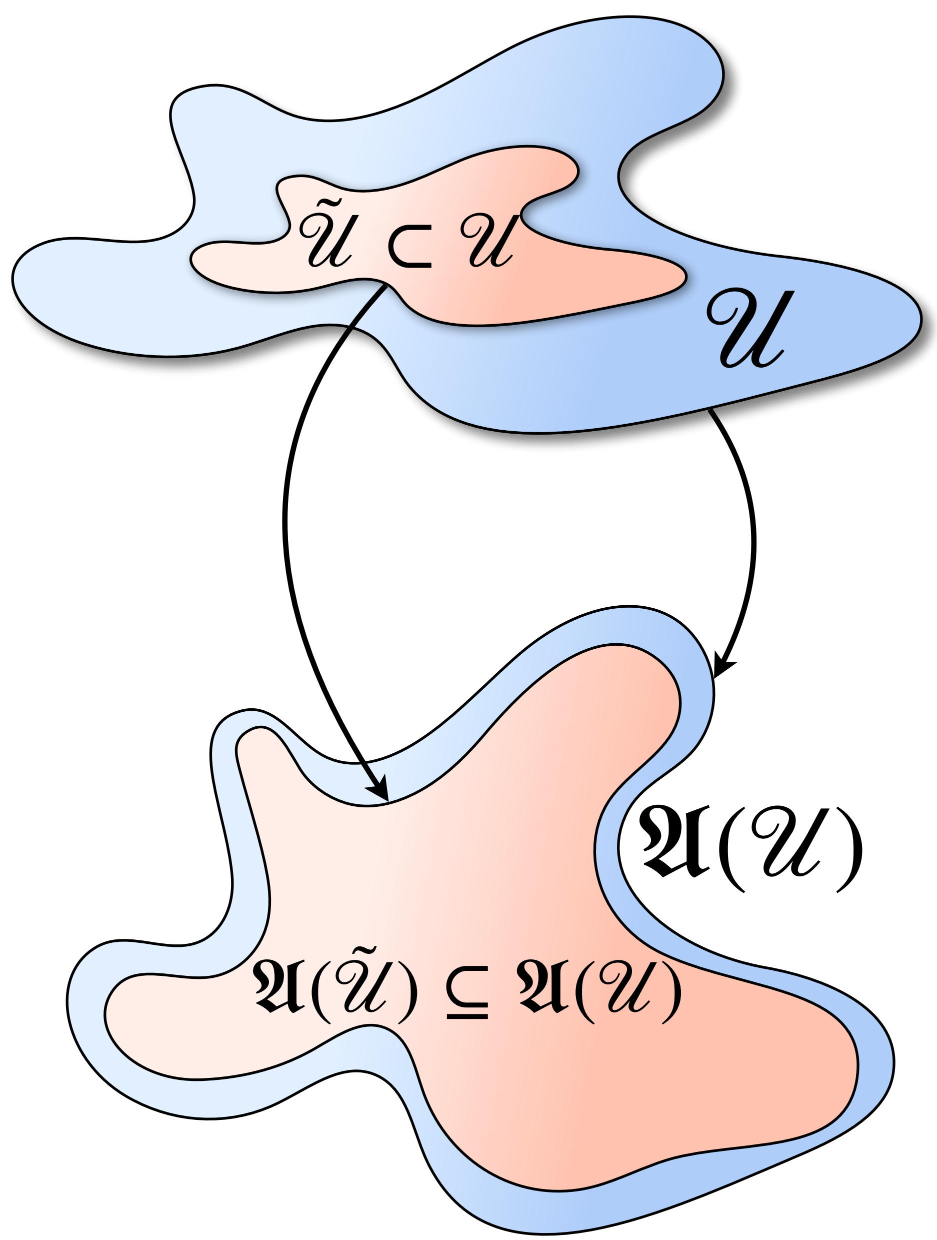

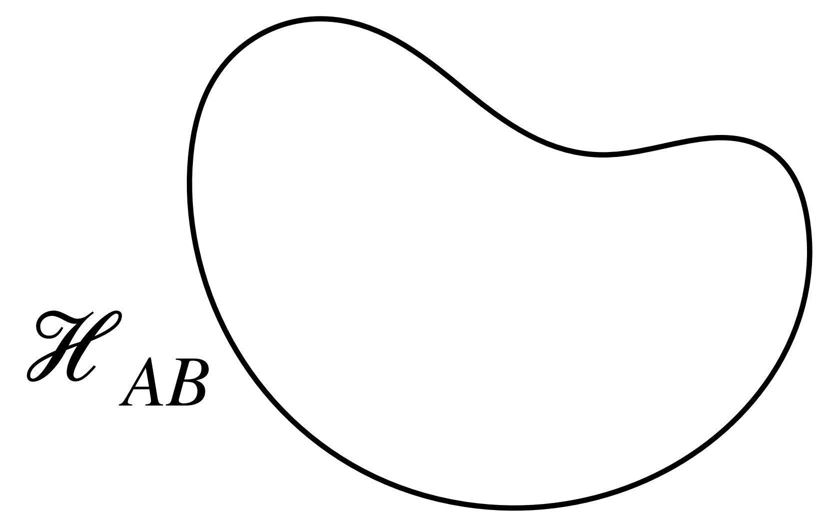

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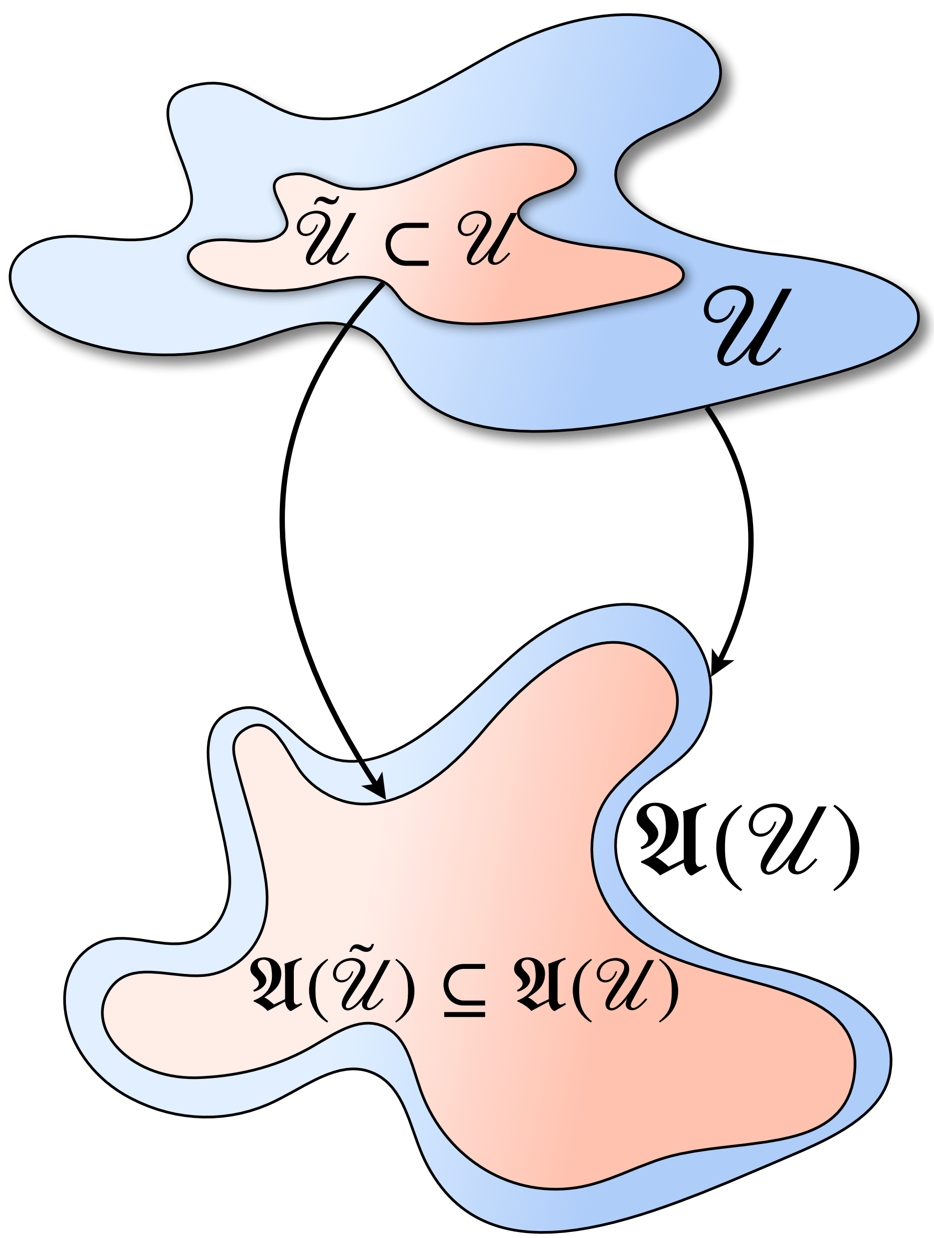

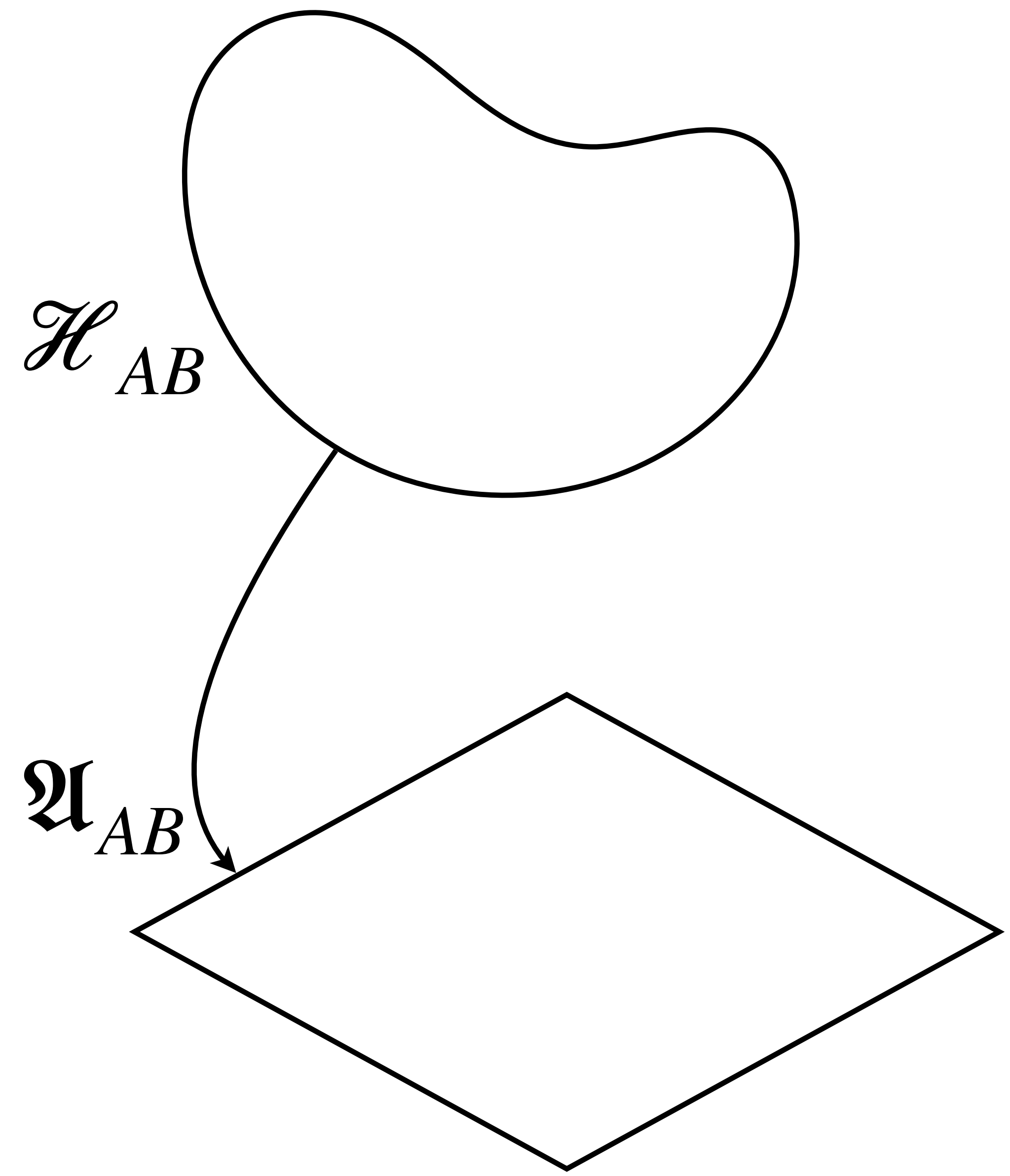

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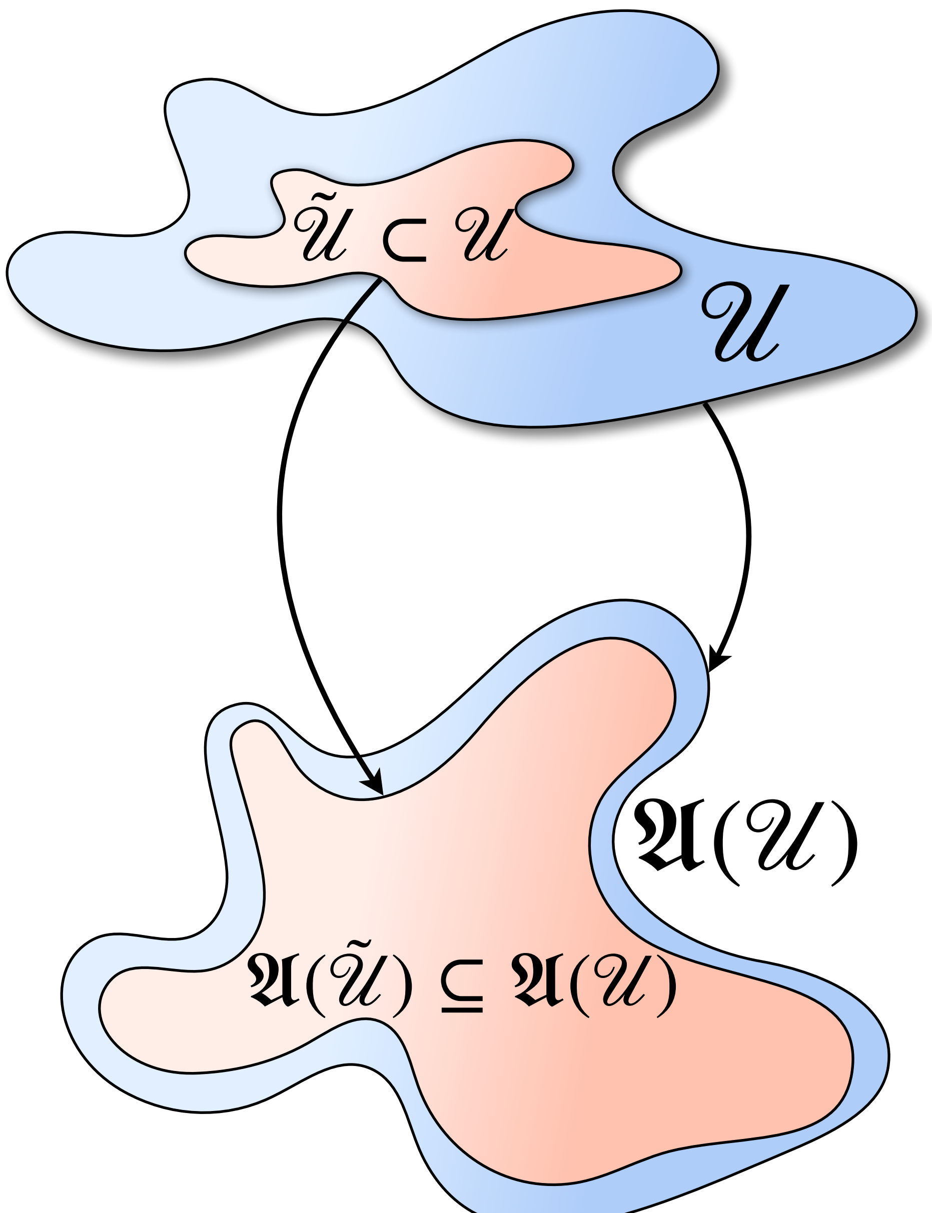

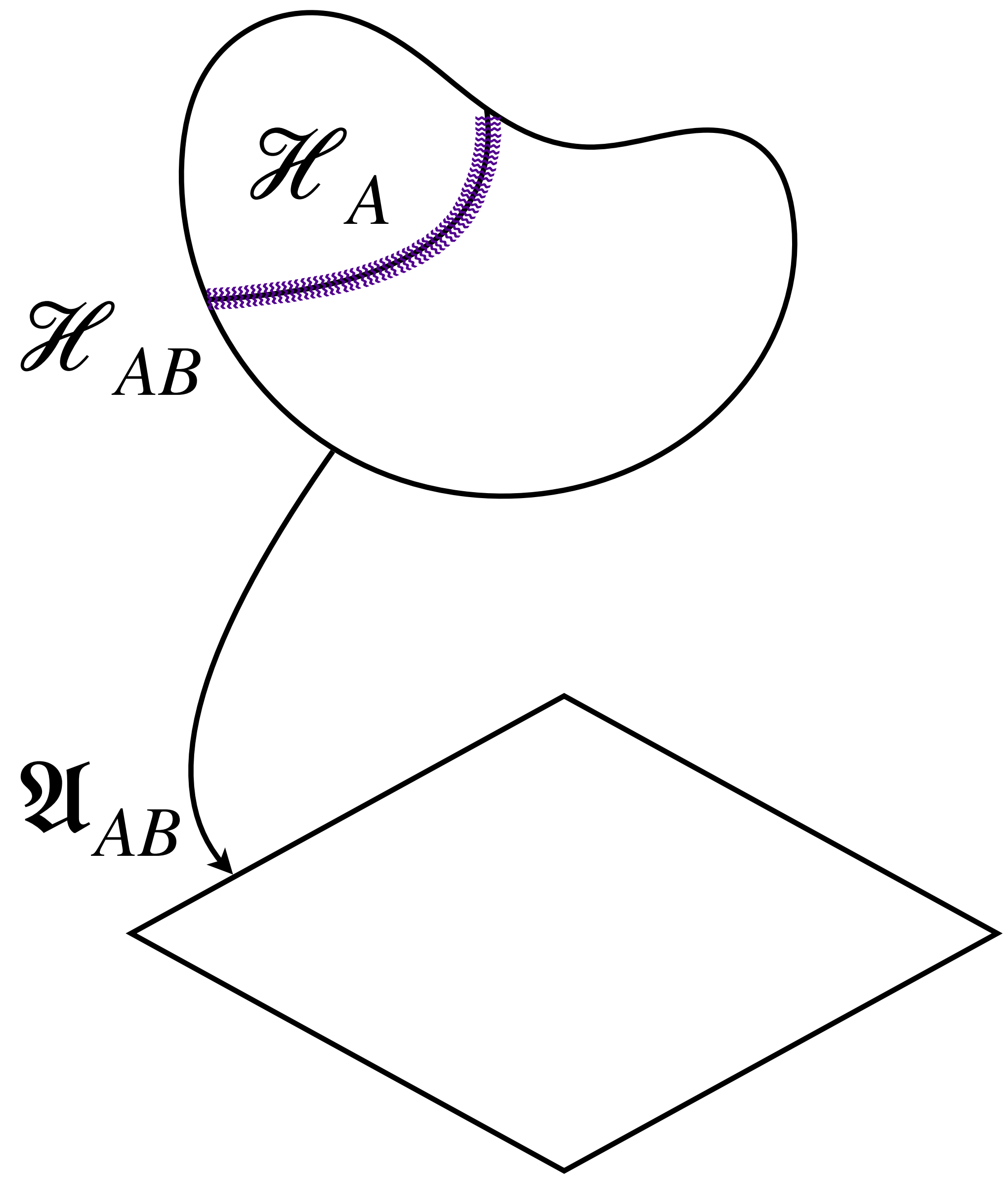

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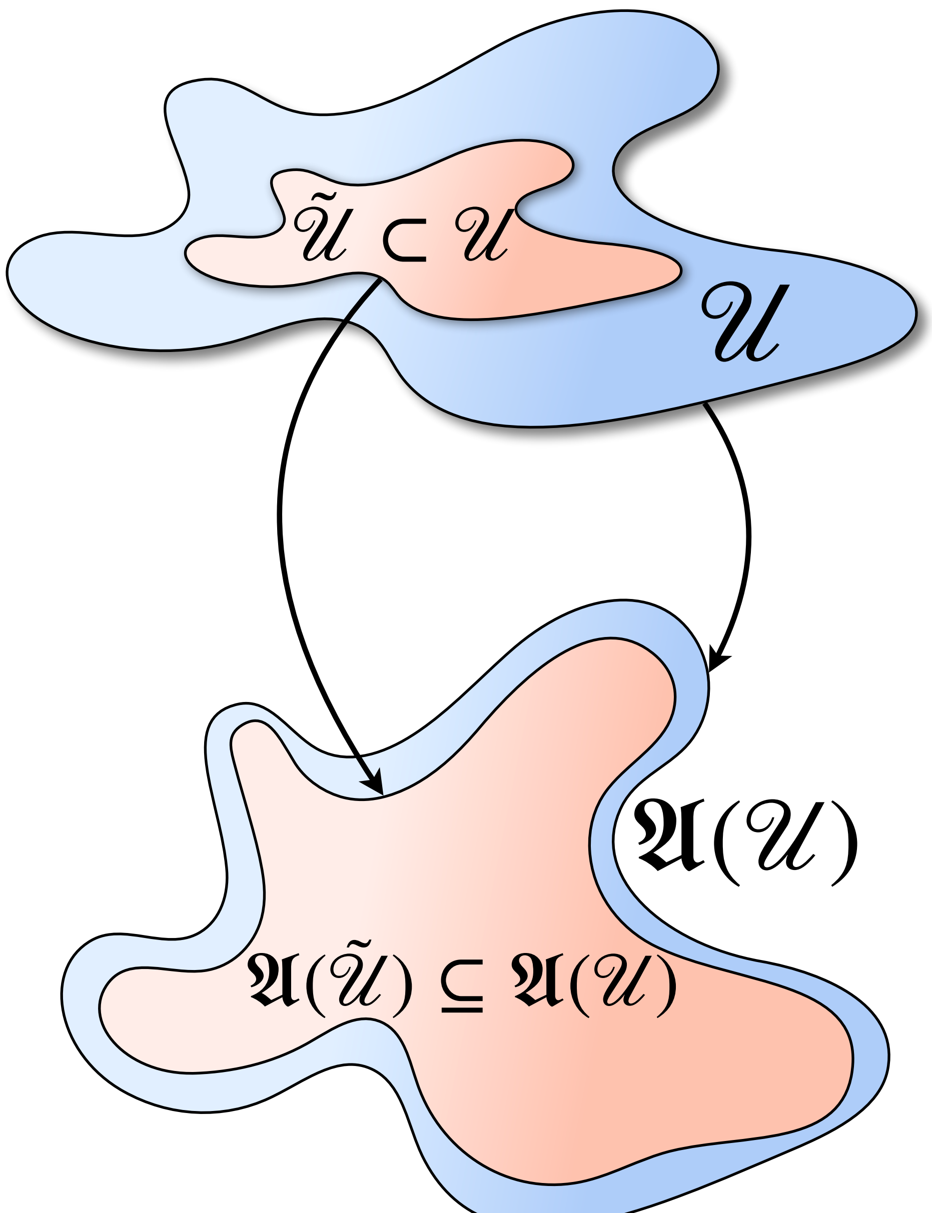

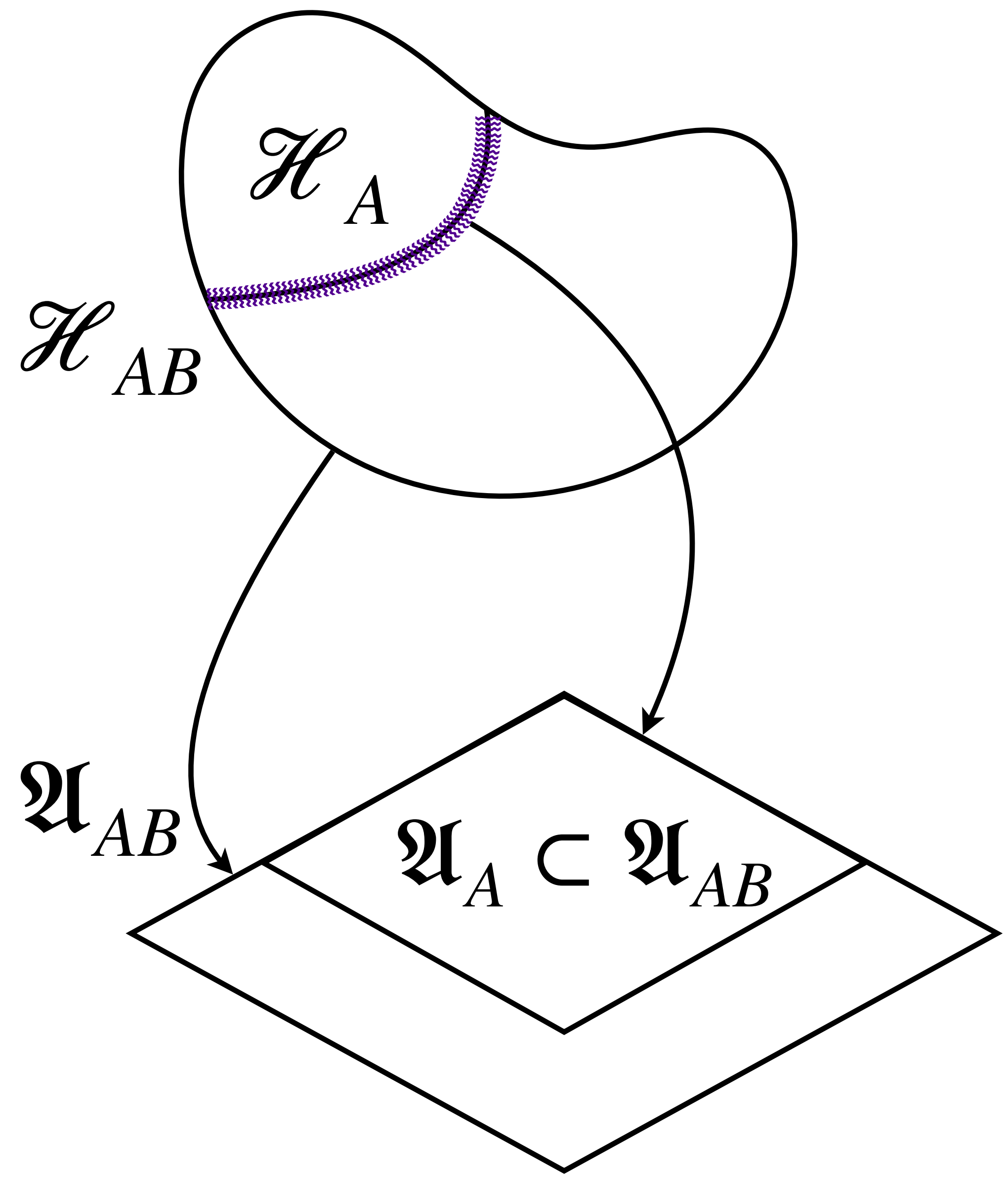

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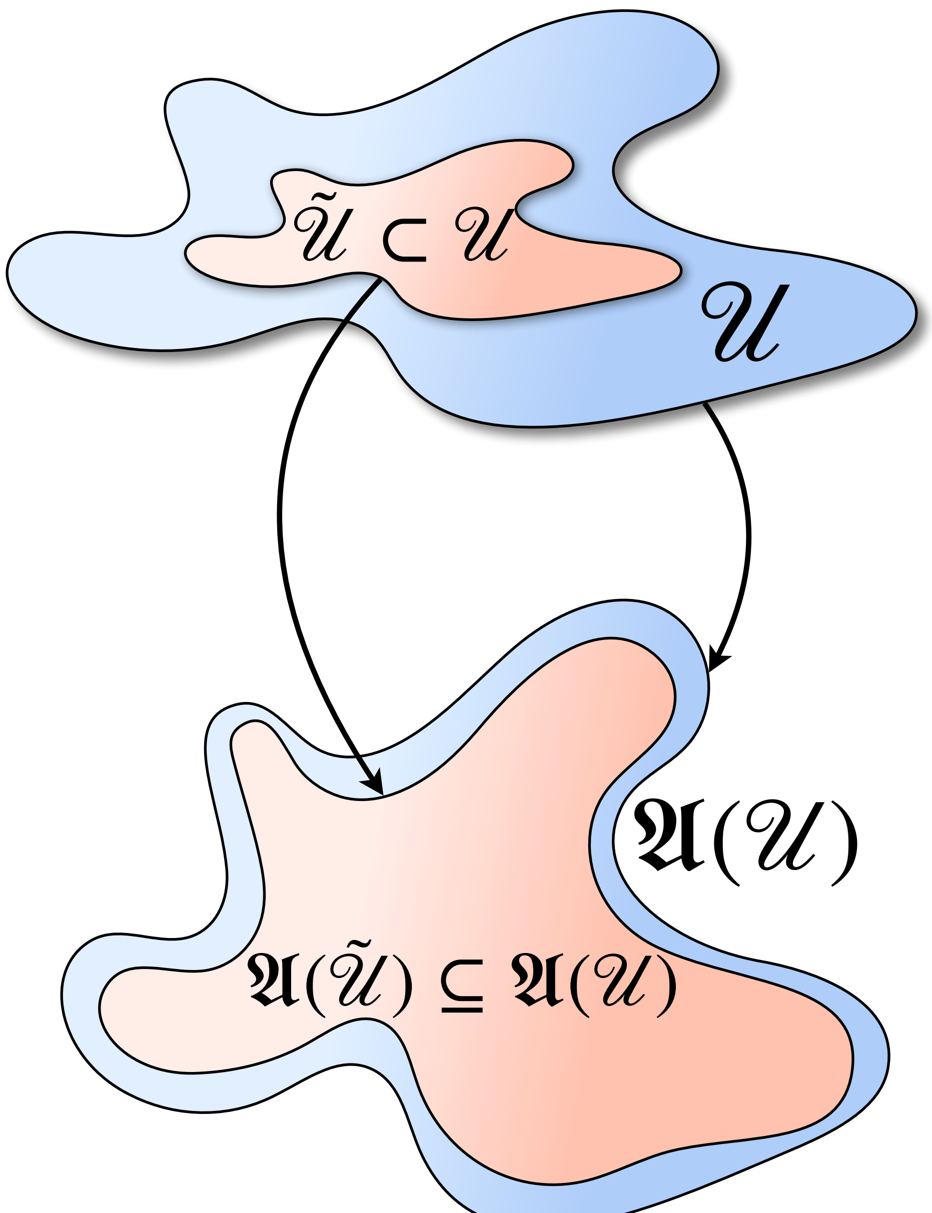
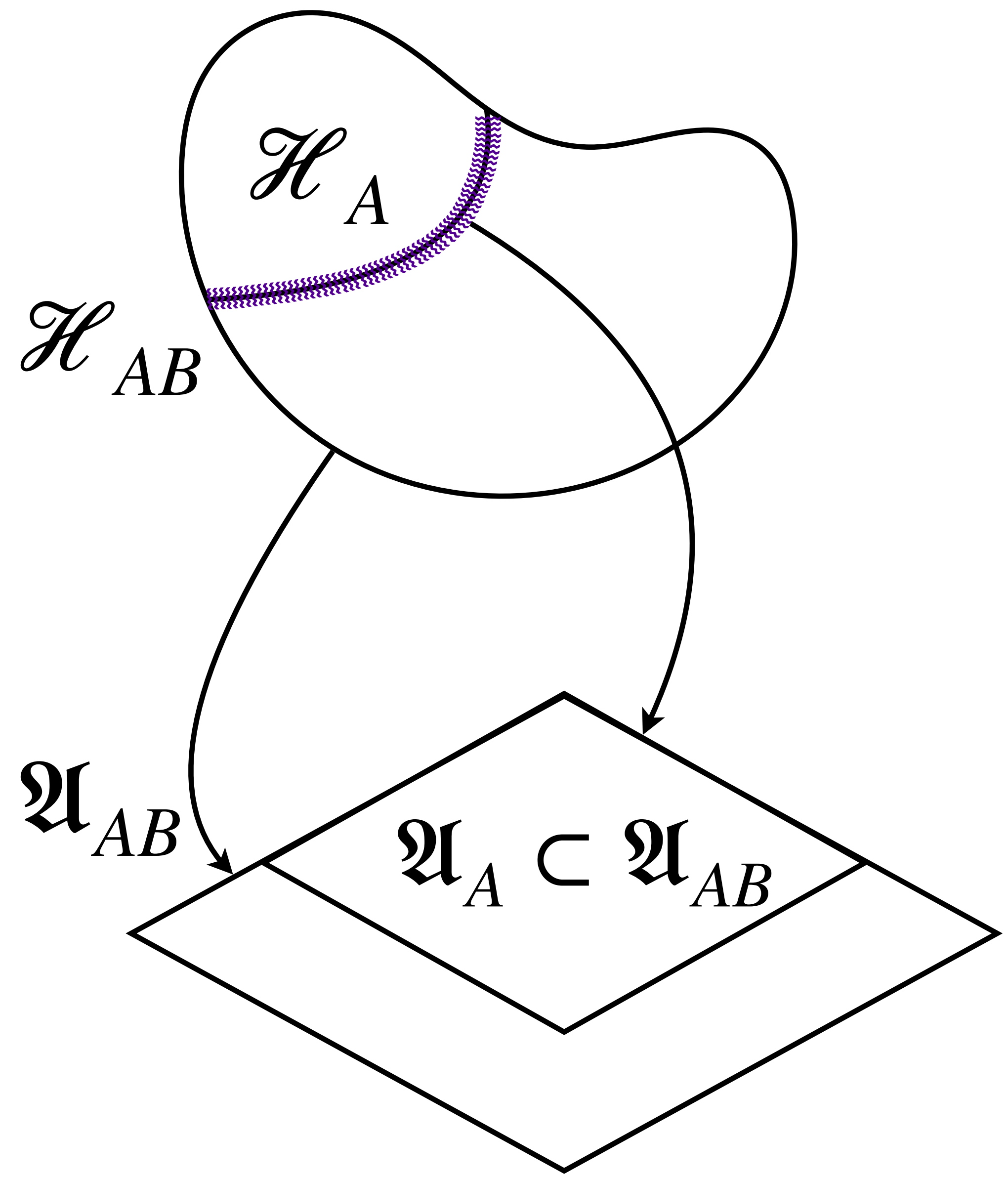




$$\varphi : \mathfrak{A}_A \to \mathfrak{A}_{AB},\ \mathbf{a} \mapsto \varphi(\mathbf{a}) = \mathbf{a} \otimes 1$$

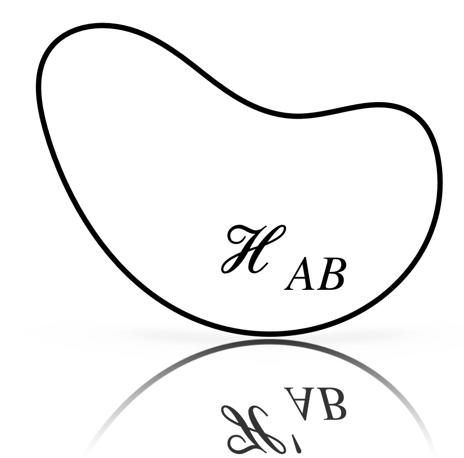
FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- To get pure state "vector", we purifies the density matrices in $\mathscr{H}_{AB}$ with a doubled Hilbert space $\mathscr{H}_{AB} \otimes \mathscr{H}'_{AB}$

- There are pure states $\Psi_{AB}, \Phi_{AB} \in \mathscr{H}_{AB} \otimes \mathscr{H}'_{AB}$ associated to the density matrices ρ_{AB} and σ_{AB}, respectively.

- We assume that ρ_{AB} is non-degenerate (otherwise one can always work in a subspace of $\mathscr{H}_{AB}$), then the vector Ψ_{AB} is a cyclic separating vector.



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- With same method, we purifies the density matrices in $\mathscr{H}_A$ with a doubled Hilbert space $\mathscr{H}_A \otimes \mathscr{H}'_A$

- There are pure states $\Psi_A, \Phi_A \in \mathscr{H}_A \otimes \mathscr{H}'_A$ associated to the density matrices ρ_A and σ_A, respectively.

- The question is: for any operator $\mathbf{a}$ acts on $\mathscr{H}_A \otimes \mathscr{H}'_A$, how to map it to an operator acts on $\mathscr{H}_{AB} \otimes \mathscr{H}'_{AB}$ naturally with a suitable isometric embedding?

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$$U: \ \mathscr{H}_A \otimes \mathscr{H}'_A \rightarrow \mathscr{H}_{AB} \otimes \mathscr{H}'_{AB}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- A natural way is keeping the factors in invariant:

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$$U(\mathbf{a}\Psi_A) = (\mathbf{a} \otimes \mathbf{1})\Psi_{AB}$$

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- A natural way is keeping the factors in invariant:

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- Because Ψ_A is cyclic, U is a linear transformation defined on the whole $\mathscr{H}_A \otimes \mathscr{H}'_A$;
- Because Ψ_A is separating, $U(0) = 0$;
- Because Ψ_{AB} is separating, U is an embedding.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- U is an isometric embedding

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- U is an isometric embedding

$$\langle U\eta \,|\, U\chi\rangle = \langle U(\mathbf{a}_\eta \Psi_A) \,|\, U(\mathbf{a}_\chi \Psi_A)\rangle = \langle (\mathbf{a}_\eta \otimes \mathbf{1})\Psi_{AB} \,|\, (\mathbf{a}_\chi \otimes \mathbf{1})\Psi_{AB}\rangle$$

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$$= \langle \Psi_{AB} \,|\, (\mathbf{a}_\eta^\dagger \mathbf{a}_\chi \otimes \mathbf{1}) \,|\, \Psi_{AB} \rangle = \mathbf{Tr}\ \rho_{AB}(\mathbf{a}_\eta^\dagger \mathbf{a}_\chi \otimes \mathbf{1})$$

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$$= \mathbf{Tr}\ \rho_A \mathbf{a}_\eta^\dagger \mathbf{a}_\chi = \langle \Psi_A \,|\, \mathbf{a}_\eta^\dagger \mathbf{a}_\chi \,|\, \Psi_A\rangle = \langle \mathbf{a}_\eta \Psi_A \,|\, \mathbf{a}_\chi \Psi_A\rangle$$

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$$= \langle \eta \,|\, \chi\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- U commutes with the action of $\mathfrak{A}_A$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

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$$\begin{array}{ccc} \mathscr{H}_A \otimes \mathscr{H}'_A & \xrightarrow{\quad \mathbf{a} \quad} & \mathscr{H}_A \otimes \mathscr{H}'_A \\ & & \Big\downarrow U \\ & & \mathscr{H}_{AB} \otimes \mathscr{H}'_{AB} \end{array}$$

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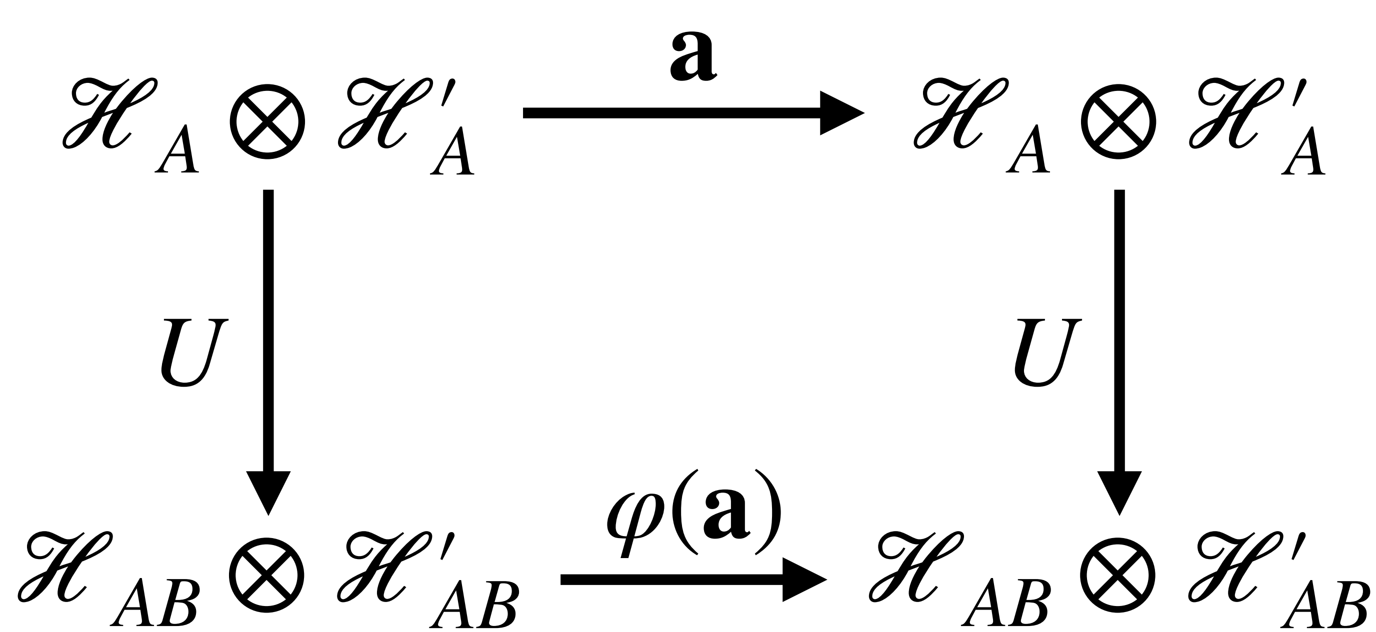

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&= \langle \Phi_A \,|\, \mathbf{b}\mathbf{a}^\dagger \,|\, \Phi_A \rangle = \mathbf{Tr}\ \sigma_A(\mathbf{b}\mathbf{a}^\dagger)
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$$= \langle (\mathbf{a} \otimes \mathbf{1})\Psi_{AB} | S_{AB}^\dagger S_{AB} | (\mathbf{b} \otimes \mathbf{1})\Psi_{AB} \rangle$$
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&= \langle S_{AB}(\mathbf{b} \otimes \mathbf{1})\Psi_{AB} | S_{AB}(\mathbf{a} \otimes \mathbf{1})\Psi_{AB} \rangle \\
&= \langle (\mathbf{b}^\dagger \otimes \mathbf{1})\Phi_{AB} | (\mathbf{a}^\dagger \otimes \mathbf{1})\Phi_{AB} \rangle = \langle \Phi_{AB} | (\mathbf{b} \otimes \mathbf{1})(\mathbf{a}^\dagger \otimes \mathbf{1}) | \Phi_{AB} \rangle \\
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- Because we have proved $\log(U^\dagger X U) \geqslant U^\dagger (\log X) U$ for any embedding U, we have
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$$\therefore \;\; \mathcal{S}(\rho_A \| \sigma_A) = -\langle \Psi_A | \log \Delta_A | \Psi_A \rangle \leqslant -\langle \Psi_A | U^\dagger(\log \Delta_{AB})U | \Psi_A \rangle$$

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&= -\langle U\Psi_A | \log \Delta_{AB} | U\Psi_A \rangle = -\langle \Psi_{AB} | \log \Delta_{AB} | \Psi_{AB} \rangle \\
&= \mathcal{S}(\rho_{AB} \| \sigma_{AB})
\end{aligned}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- Because $U^\dagger \Delta_{AB} U = \Delta_A$, the key point in the proof is that the logarithm function satisfies $\log(U^\dagger X U) \geqslant U^\dagger (\log X) U$ for any embedding U.

- So one may replace $\log X$ with other functions which are increasing function of a positive operator X.

- An example is X^α, $0 \leqslant \alpha \leqslant 1$

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$$\langle \Psi_A | \Delta_A^\alpha | \Psi_A \rangle \geqslant \langle \Psi_{AB} | \Delta_{AB}^\alpha | \Psi_{AB} \rangle$$

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$$\langle \Psi_A | \Delta_A^\alpha | \Psi_A \rangle \geqslant \langle \Psi_{AB} | \Delta_{AB}^\alpha | \Psi_{AB} \rangle$$

$$\therefore \;\; \mathbf{Tr}_A \sigma_A^\alpha \rho_A^{1-\alpha} \geqslant \mathbf{Tr}_{AB} \sigma_{AB}^\alpha \rho_{AB}^{1-\alpha}, \qquad 0 \leqslant \alpha \leqslant 1$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- When $\alpha = 0$, $\mathbf{Tr}_A \sigma_A^\alpha \rho_A^{1-\alpha} = \mathbf{Tr}_{AB} \sigma_{AB}^\alpha \rho_{AB}^{1-\alpha}$.

- Expanding around $\alpha = 0$

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- Expanding around $\alpha = 0$

$$\mathbf{Tr} \sigma^\alpha \rho^{-\alpha} \rho = \mathbf{Tr}(\mathbf{1} + \alpha \log \sigma)(\mathbf{1} - \alpha \log \rho)\rho + \mathcal{O}(\alpha^2)$$

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$$\begin{aligned}\mathbf{Tr}\sigma^\alpha \rho^{-\alpha} \rho &= \mathbf{Tr}(\mathbf{1} + \alpha \log \sigma)(\mathbf{1} - \alpha \log \rho)\rho + \mathcal{O}(\alpha^2) \\ &= 1 - \alpha \mathbf{Tr}\rho(\log \rho - \log \sigma) + \mathcal{O}(\alpha^2)\end{aligned}$$

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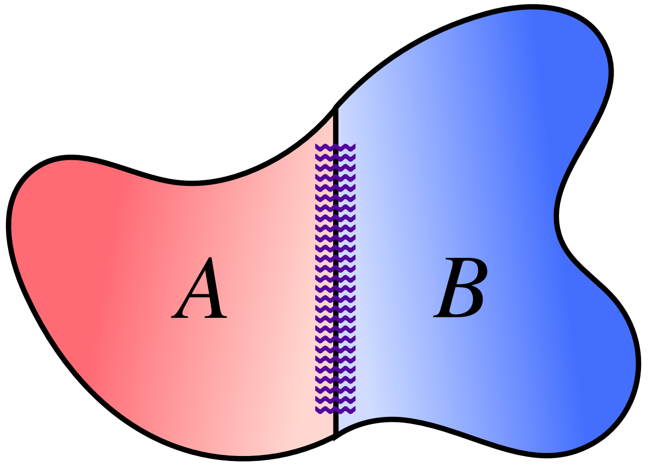
FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- Some results in quantum information theory
  - von Neumann entropy of a density matrix ρ is $\mathcal{S} = -\,\mathbf{Tr}\ \rho \log \rho$;
  - For bipartite system $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, there are reduced density matrices $\rho_A = \mathbf{Tr}_B \rho_{AB}$ and $\rho_B = \mathbf{Tr}_A \rho_{AB}$ for density matrix ρ_{AB}, one may denote $\mathcal{S}_{AB} = \mathcal{S}(\rho_{AB})$, $\mathcal{S}_A = \mathcal{S}(\rho_A)$ and $\mathcal{S}_B = \mathcal{S}(\rho_B)$;
  - The mutual information between subsystem A and B is

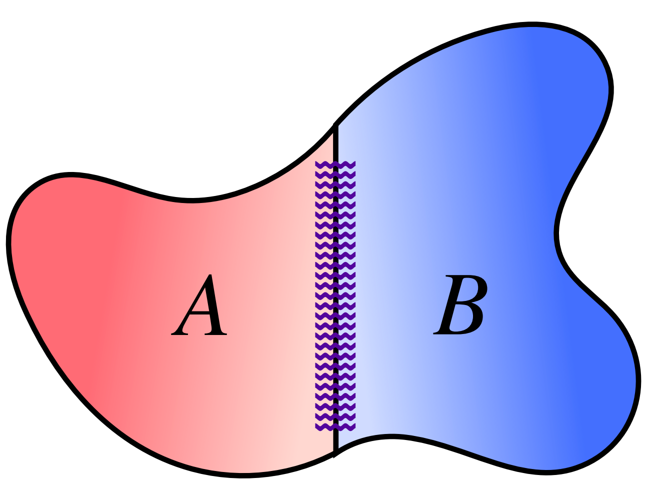
$$I(A;B) = \mathcal{S}_A + \mathcal{S}_B - \mathcal{S}_{AB}$$



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- Some results in quantum information theory
  - **Subadditivity** of quantum entropy: $I(A;B) \geqslant 0$ for all ρ_{AB};
  - Proof: define $\sigma_{AB} = \rho_A \otimes \rho_B$



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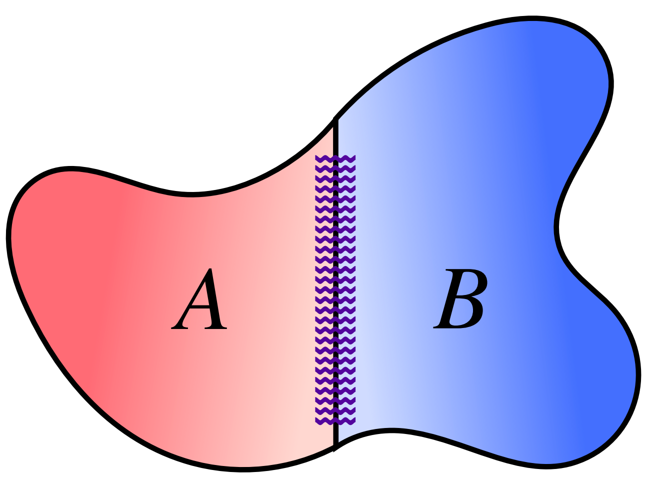

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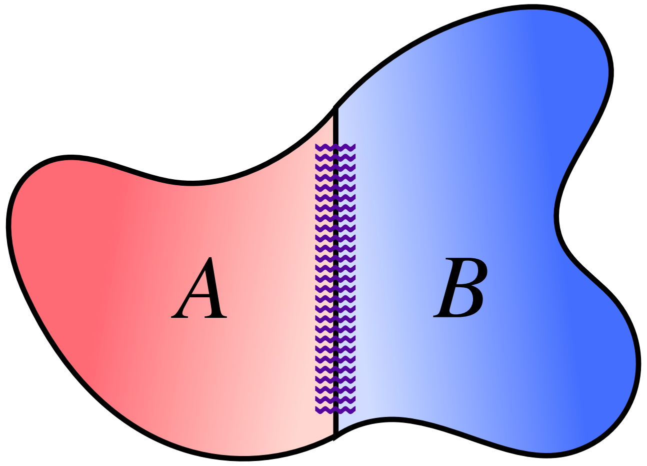

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

III. Monotonicity of relative entropy in the finite-dimensional case

- Some results in quantum information theory
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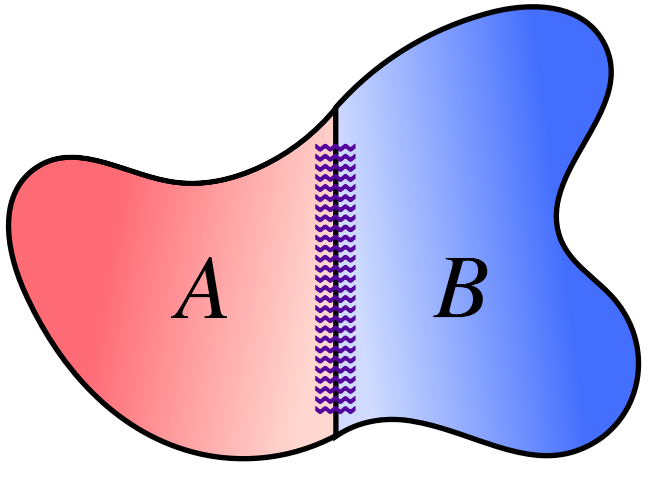

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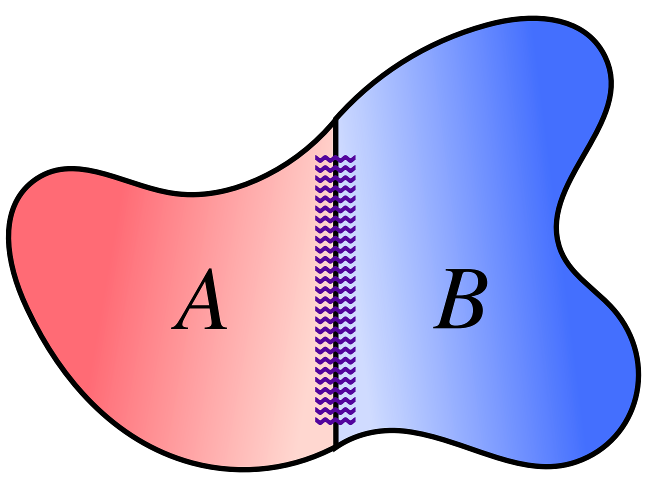

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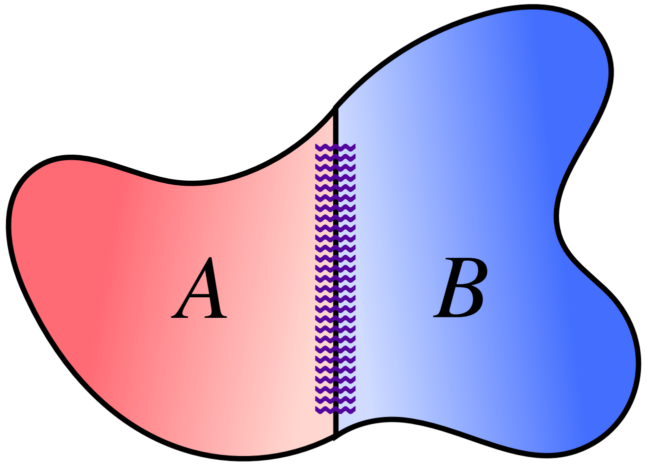

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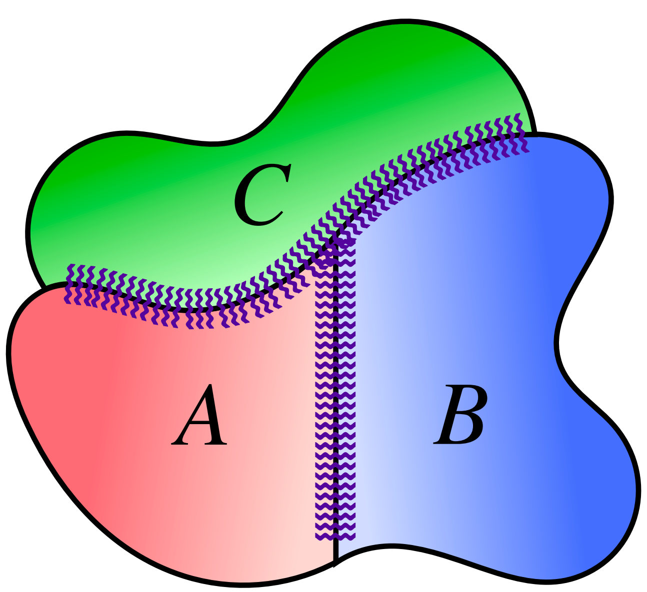

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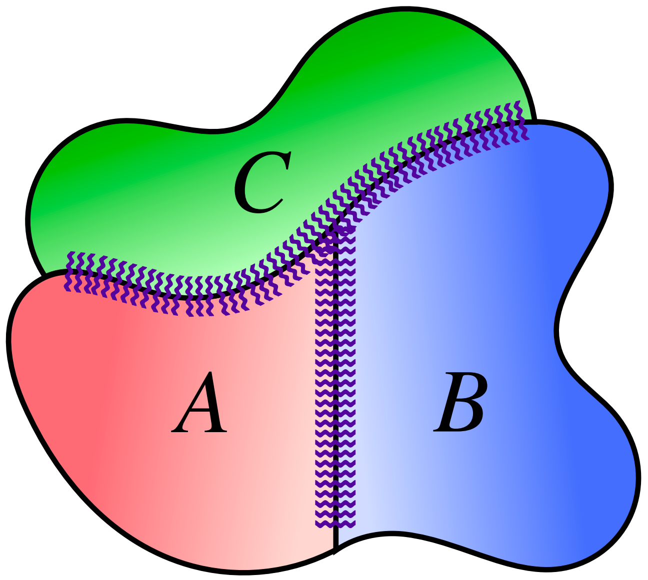

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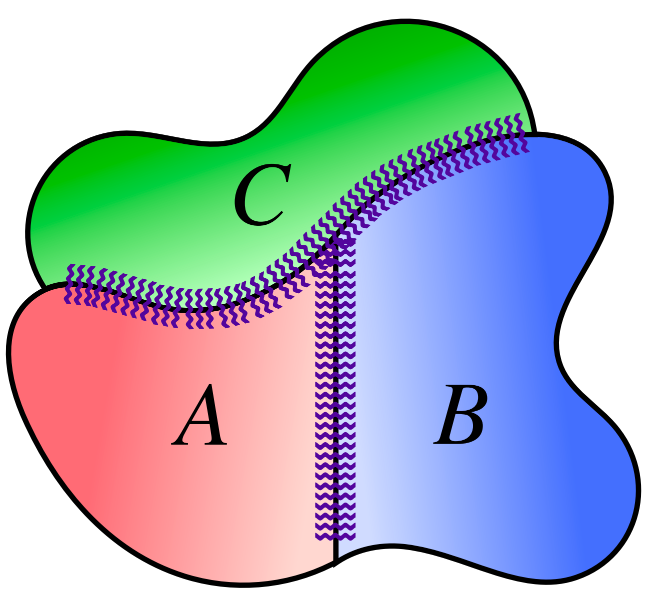

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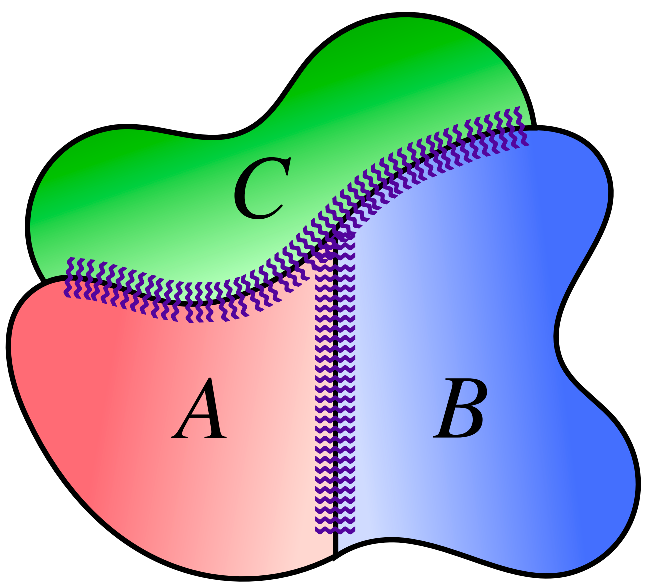

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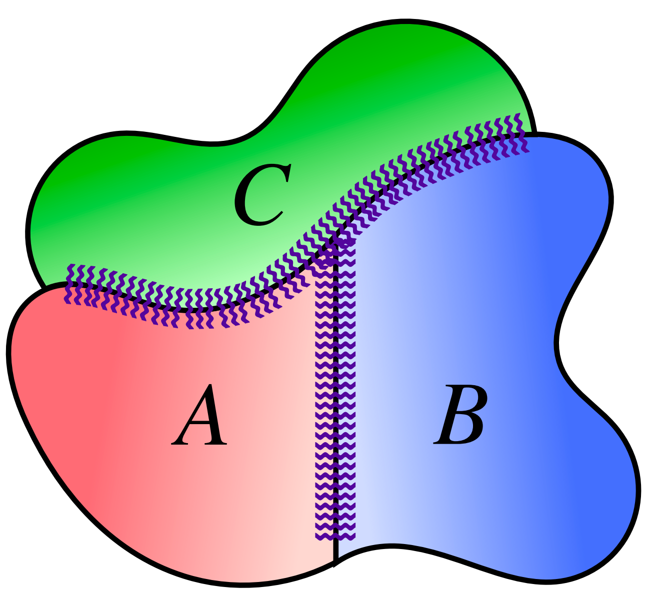

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\end{aligned}
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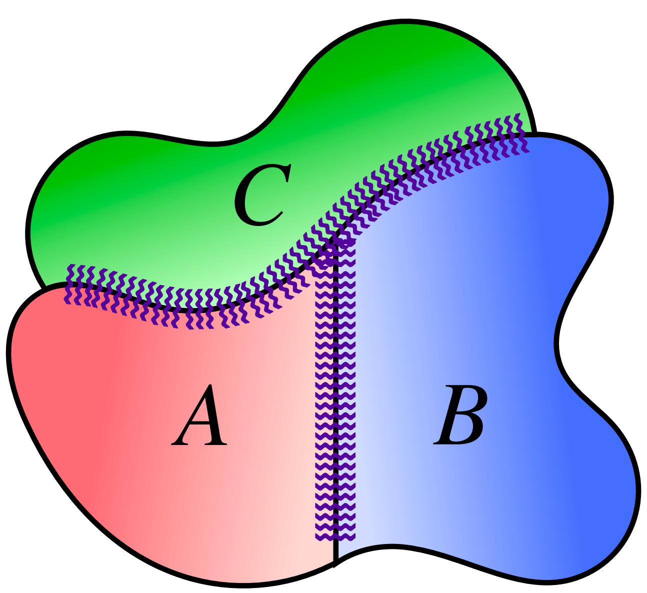

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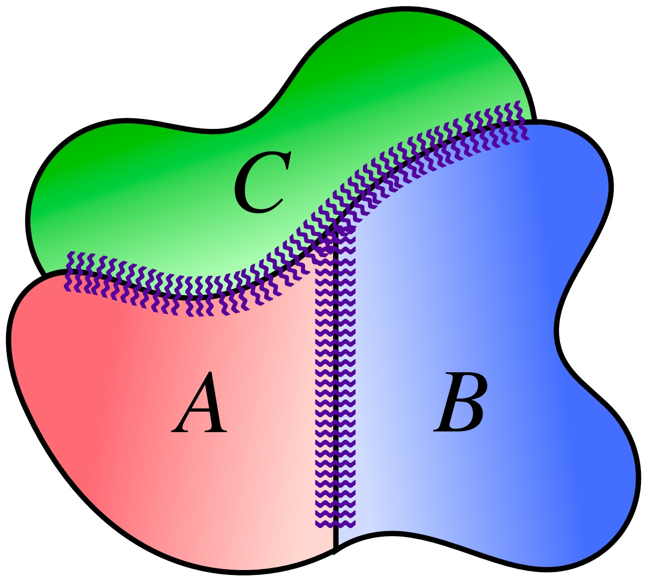

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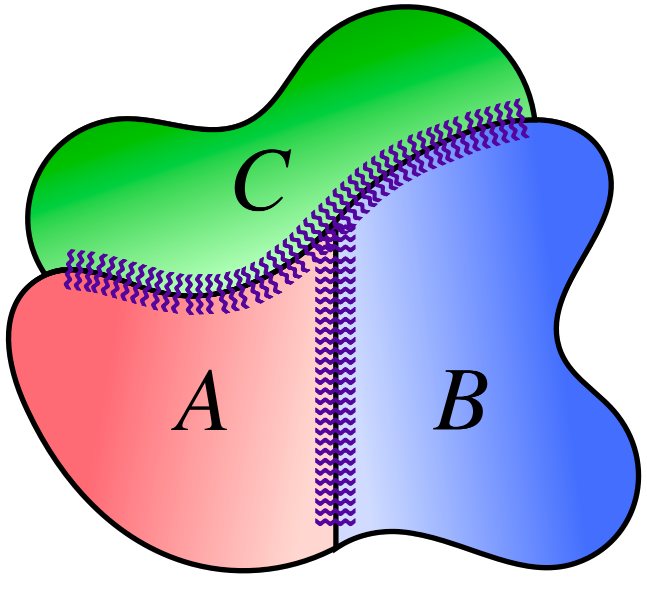

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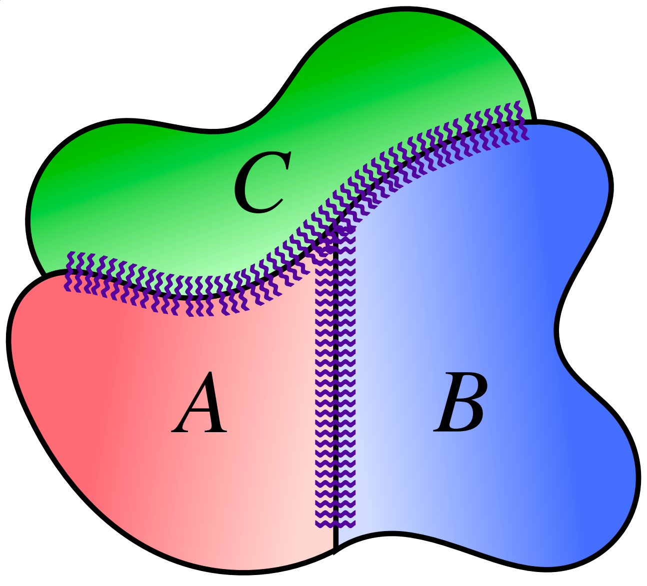
$$\mathcal{S}_{AB} + \mathcal{S}_{AC} \geqslant \mathcal{S}_{ABC} + \mathcal{S}_B$$



A FUNDAMENTAL EXAMPLE

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I. Overview

- A simple decomposition of Minkowski spacetime $\mathscr{M}_D$

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$$\mathscr{M}_D \sim \mathbb{R}^{1,1} \times \mathbb{R}^{D-2}$$

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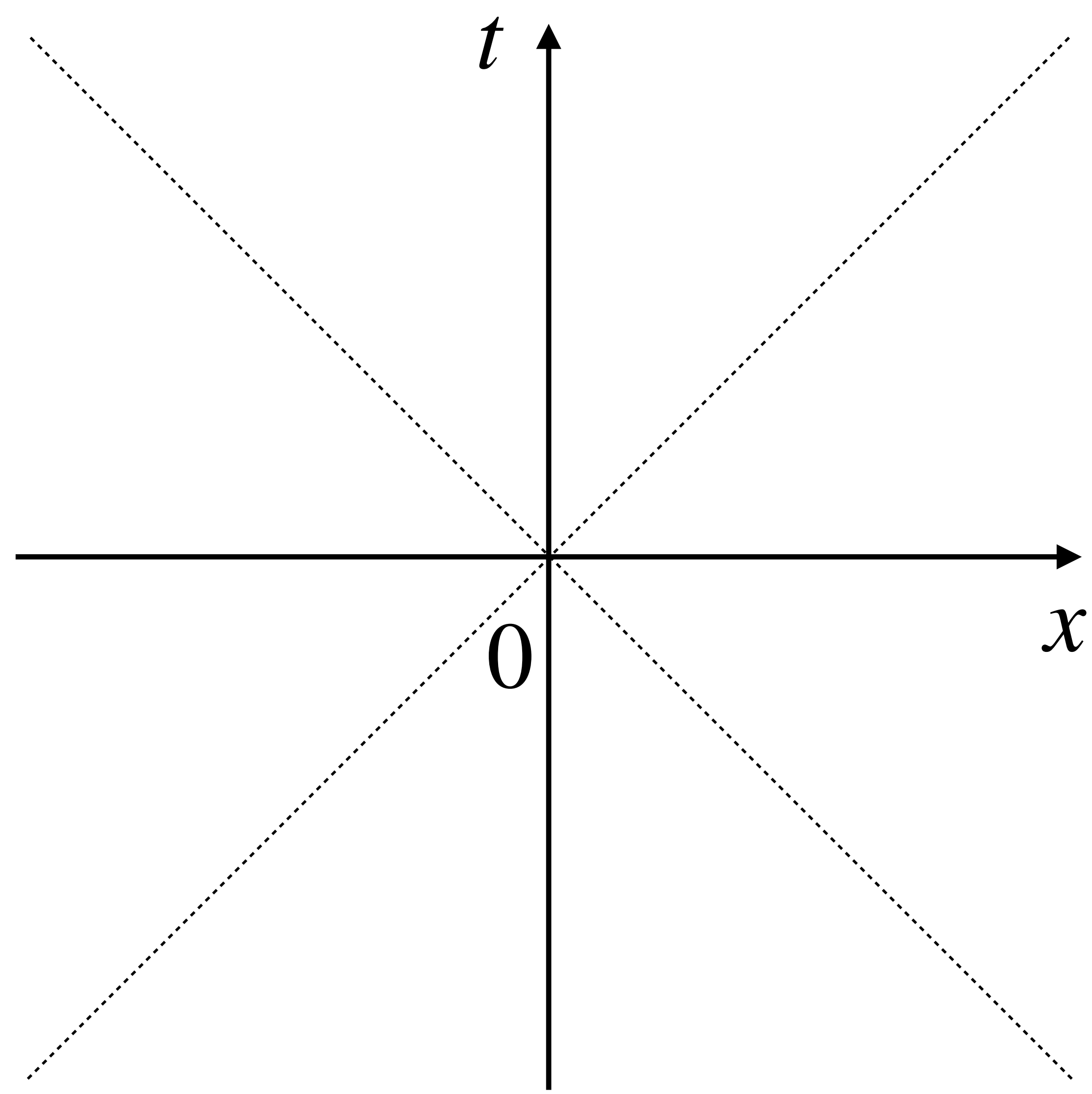

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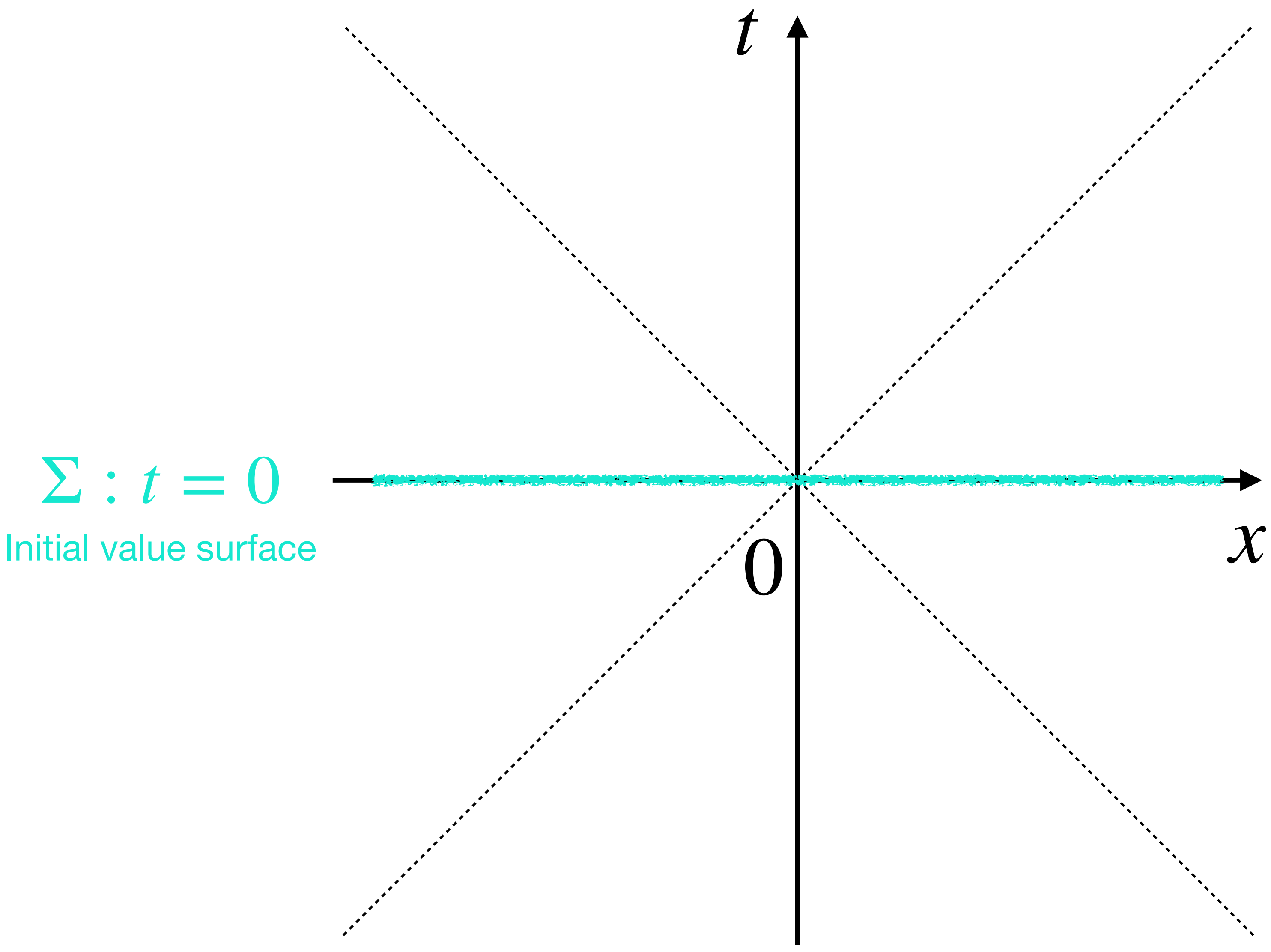

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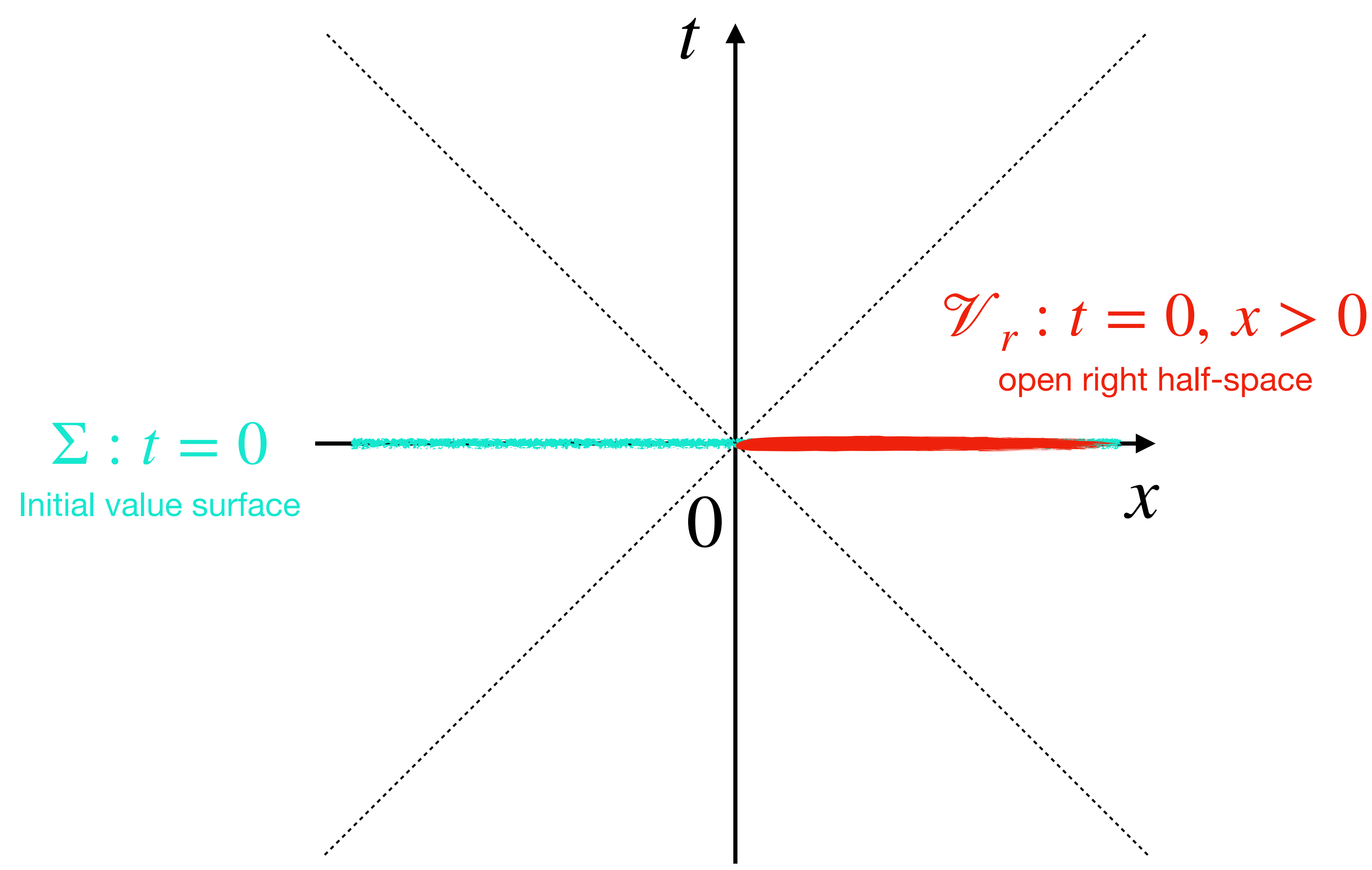

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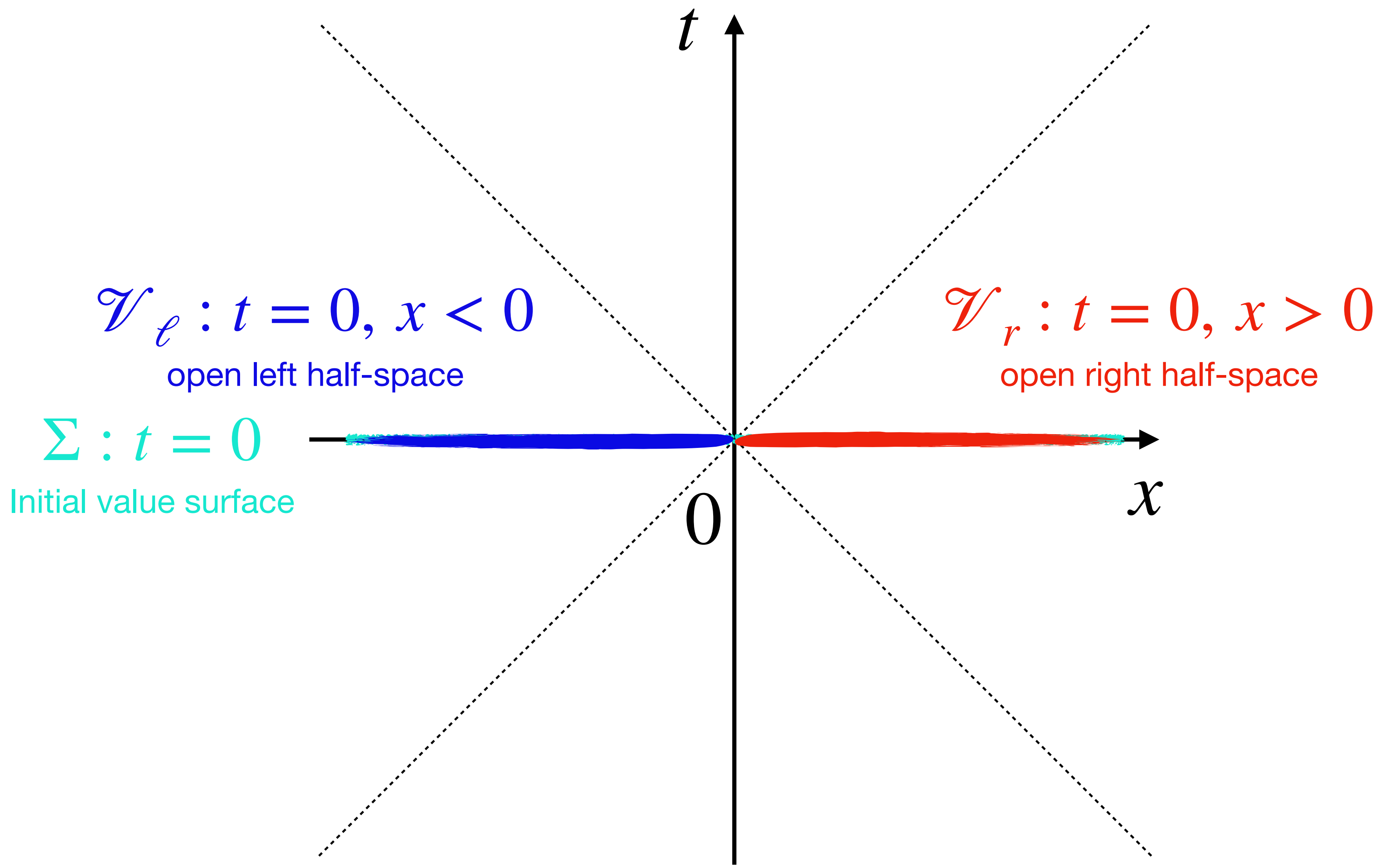

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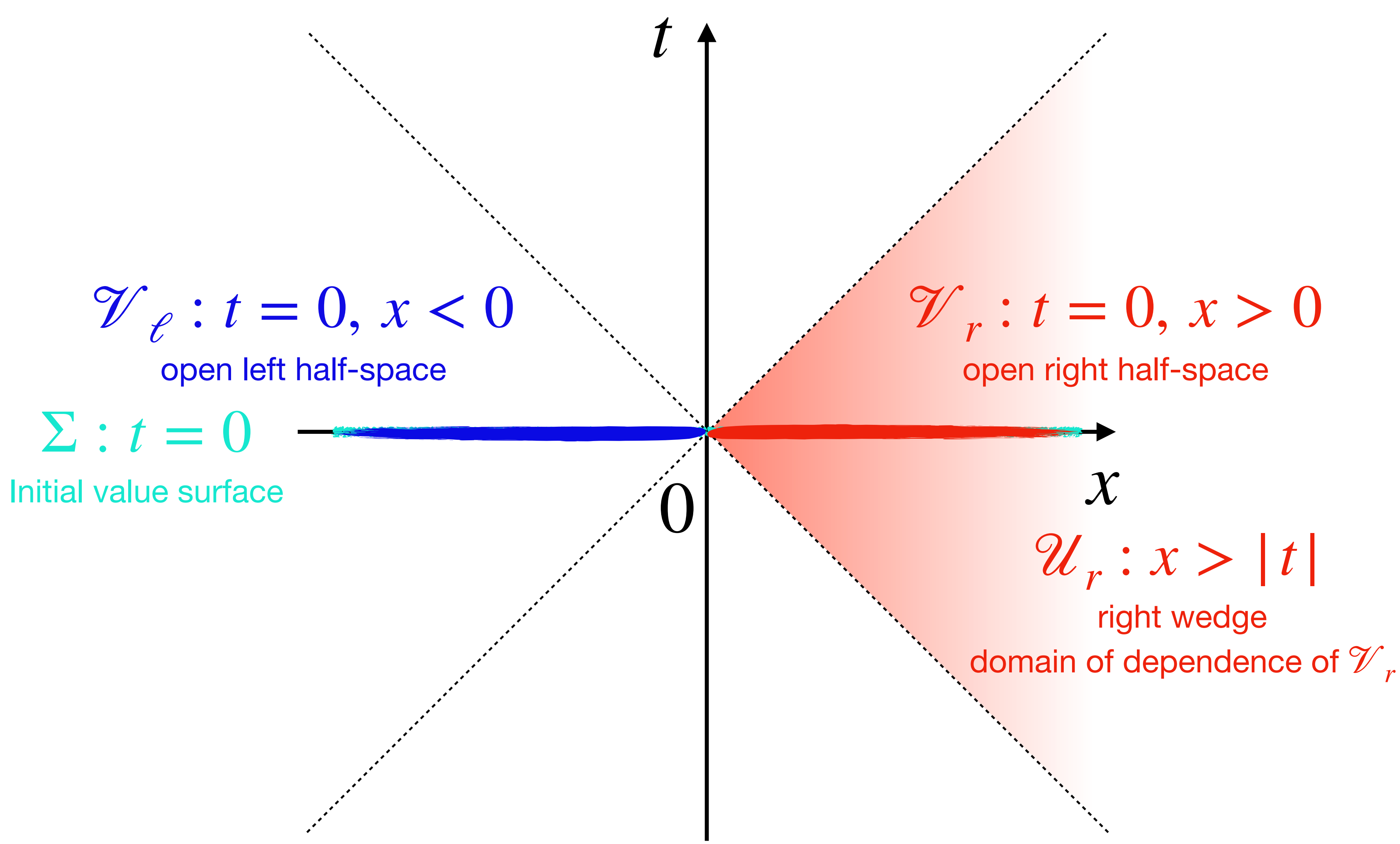

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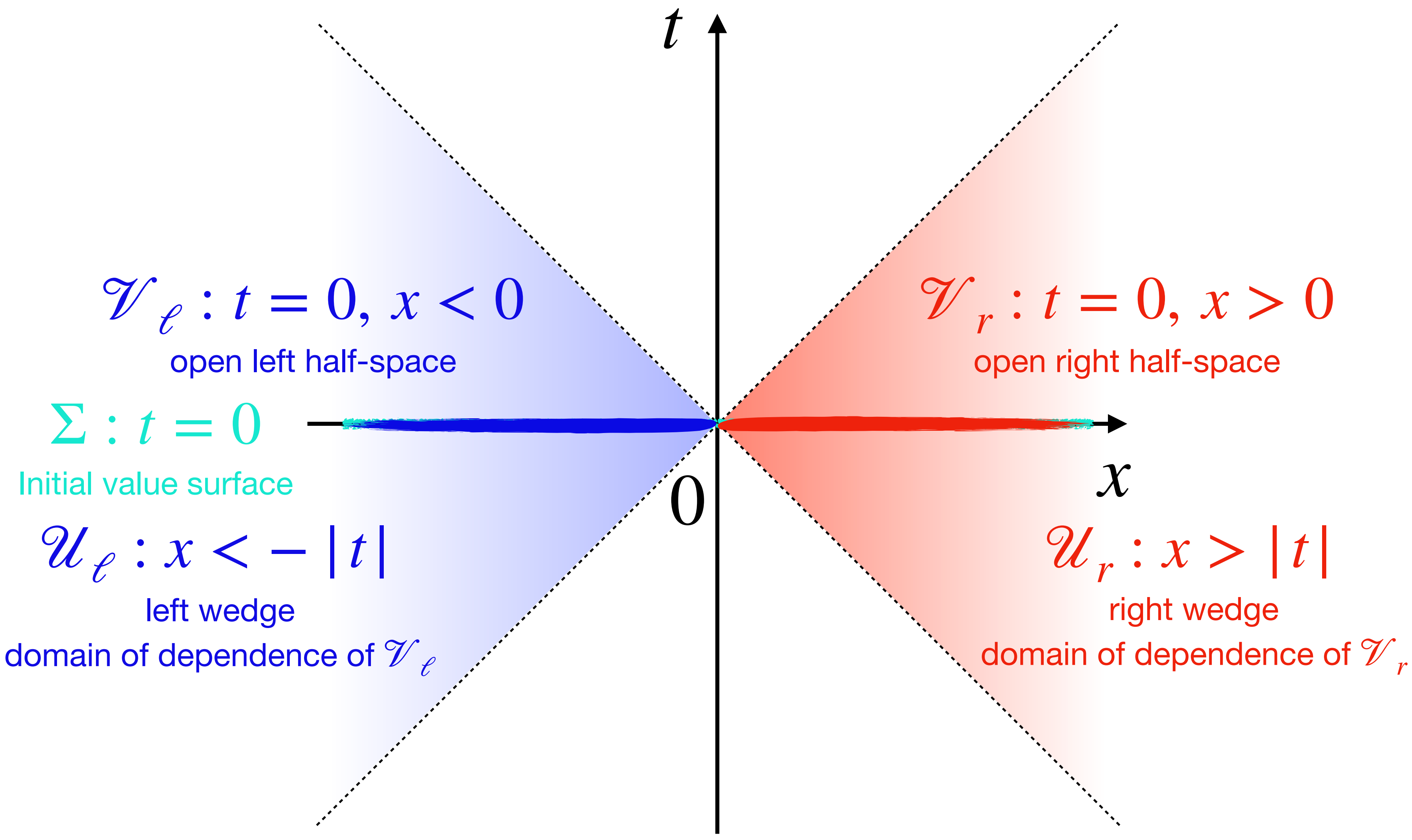

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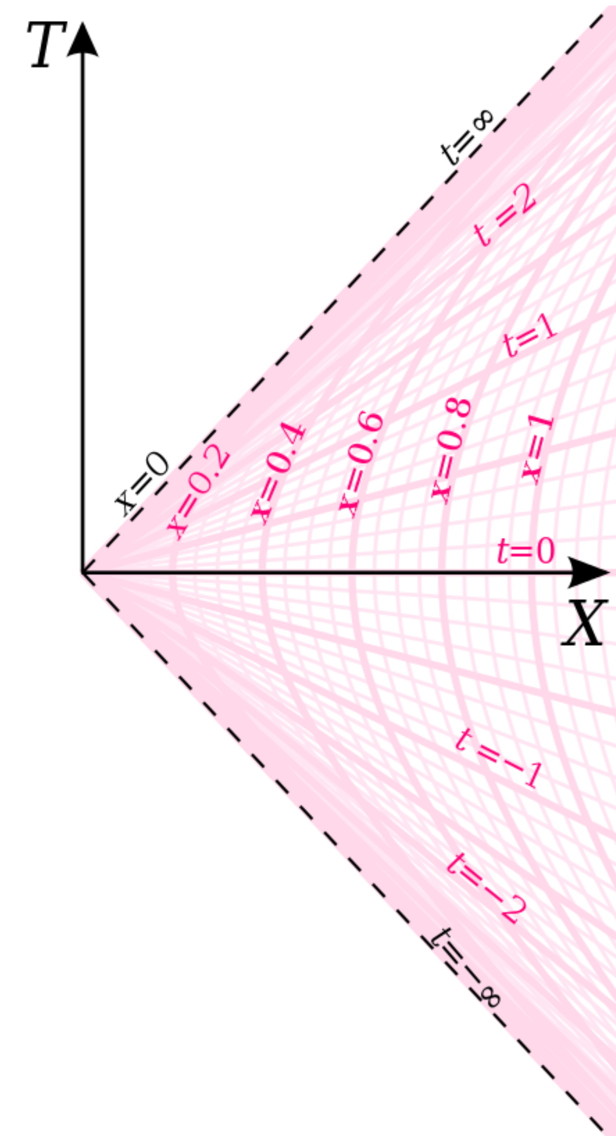
- Rindler space (Rindler, 1966)



A FUNDAMENTAL EXAMPLE

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Wolfgang Rindler
(1924/05/18-2019/02/08)

A FUNDAMENTAL EXAMPLE

I. Overview

- A simple decomposition of Minkowski spacetime $\mathscr{M}_D$
- Rindler space
- The local observable algebra associated with the right (left) wedge $\mathscr{U}_r$ ($\mathscr{U}_\ell$) is denoted as $\mathfrak{A}_r$ ($\mathfrak{A}_\ell$).
- $\mathfrak{A}_r \subseteq \mathfrak{A}'_\ell$, we will learn later that $\mathfrak{A}_r = \mathfrak{A}'_\ell$.
- Let Ω be the vacuum state of a quantum field theory on $\mathscr{M}_D$, we will determine the modular operators Δ_Ω and J_Ω for observations in region $\mathscr{U}_r$.
- (We do not use Carter-Penrose diagram here, because for Minkowski spacetime, a point in the diagram means $\mathbb{S}^{D-2}$ but not $\mathbb{R}^{D-2}$.)

A FUNDAMENTAL EXAMPLE

I. Overview

- The modular operators Δ_Ω and J_Ω for observations in region $\mathscr{U}_r$. (Wichmann and Bisognano, 1976)

Eyvind Hugo Wichmann (1928/05/30-2019/02/16)

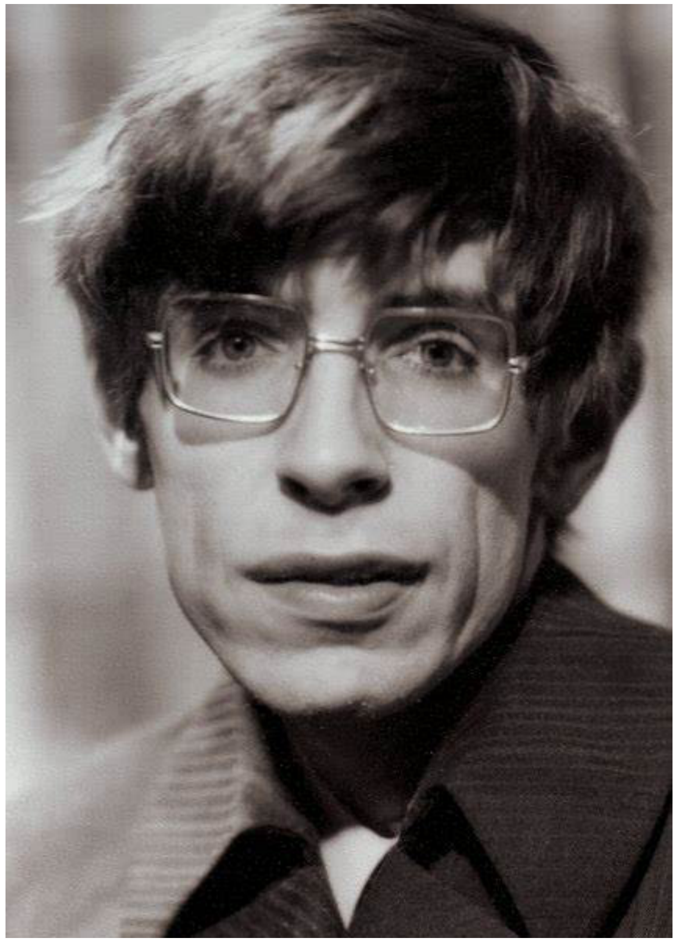
Joseph Bisognano (~1947-)

A FUNDAMENTAL EXAMPLE

I. Overview

- A direct path integral approach for this problem is important in both Unruh effect (Unruh, 1976) and Hawking radiation (Hawking, 1975, 1977)

William George "Bill" Unruh
(1945/08/28-)



Stephen William Hawking
(1942/01/08-2018/03/14)

A FUNDAMENTAL EXAMPLE

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- The CPT symmetry of quantum field theory (in flat spacetime)
- Let $\xi_{(\alpha)(\dot{\beta})} = \xi_{\alpha_1 \cdots \alpha_j \dot{\beta}_1 \cdots \dot{\beta}_k}$ and $\eta_{(\dot{\alpha})(\beta)} = \eta_{\dot{\alpha}_1 \cdots \dot{\alpha}_j \beta_1 \cdots \beta_k}$ are complex representation vector of Lorentz group

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$$U(P)\begin{pmatrix}\xi_{(\alpha)(\dot\beta)}\\ \eta_{(\dot\alpha)(\beta)}\end{pmatrix}U(P)^{-1} \to \begin{pmatrix} i & \text{if } j+k \text{ is odd} \\ 1 & \text{if } j+k \text{ is even}\end{pmatrix}\begin{pmatrix} 0 & (-1)^j\zeta\otimes\cdots\otimes\zeta \\ (-1)^k\zeta\otimes\cdots\otimes\zeta & 0\end{pmatrix}\begin{pmatrix}\xi_{(\alpha)(\dot\beta)}\\ \eta_{(\dot\alpha)(\beta)}\end{pmatrix}$$

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$$U(C)\begin{pmatrix}\xi_{(\alpha)(\dot\beta)}\\ \eta_{(\dot\alpha)(\beta)}\end{pmatrix}U(C)^{-1} \to \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi^*_{(\alpha)(\dot\beta)}\\ \eta^*_{(\dot\alpha)(\beta)}\end{pmatrix}$$

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$$U(T)\begin{pmatrix}\xi_{(\alpha)(\dot{\beta})} \\ \eta_{(\dot{\alpha})(\beta)}\end{pmatrix} U(T)^{-1} \to \begin{pmatrix} \zeta \otimes \cdots \otimes \zeta & 0 \\ 0 & \zeta \otimes \cdots \otimes \zeta \end{pmatrix} \begin{pmatrix}\xi_{(\alpha)(\dot{\beta})} \\ \eta_{(\dot{\alpha})(\beta)}\end{pmatrix}, \qquad \zeta_{\alpha\beta} = \zeta_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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$$U(T)\begin{pmatrix} \xi_{(\alpha)(\dot{\beta})} \\ \eta_{(\dot{\alpha})(\beta)} \end{pmatrix} U(T)^{-1} \to \begin{pmatrix} \zeta \otimes \cdots \otimes \zeta & 0 \\ 0 & \zeta \otimes \cdots \otimes \zeta \end{pmatrix} \begin{pmatrix} \xi_{(\alpha)(\dot{\beta})} \\ \eta_{(\dot{\alpha})(\beta)} \end{pmatrix}, \qquad \zeta_{\alpha\beta} = \zeta_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Theta \begin{pmatrix} \xi_{(\alpha)(\dot{\beta})} \\ \eta_{(\dot{\alpha})(\beta)} \end{pmatrix} \Theta^{-1} \to \begin{pmatrix} i & \text{if } j+k \text{ is odd} \\ 1 & \text{if } j+k \text{ is even} \end{pmatrix} \begin{pmatrix} (-1)^j & 0 \\ 0 & (-1)^k \end{pmatrix} \begin{pmatrix} \xi^*_{(\alpha)(\dot{\beta})} \\ \eta^*_{(\dot{\alpha})(\beta)} \end{pmatrix}$$

A FUNDAMENTAL EXAMPLE

I. Overview

- The CPT symmetry of quantum field theory (in flat spacetime)

- CPT Theorem: for Lorentz covariant quantum fields $\varphi_\mu, \cdots, \psi_\nu$, if the weak local condition (WLC)

$$\langle \Omega \,|\, \varphi_\mu(x_1) \cdots \psi_\nu(x_n) \,|\, \Omega \rangle = i^F \langle \Omega \,|\, \psi_\nu(x_n) \cdots \varphi_\mu(x_1) \,|\, \Omega \rangle$$

  holds in a (real) neighborhood of a Jost point $(x_1 - x_2, \cdots, x_{n-1} - x_n)$, then the CPT condition

$$\langle \Omega \,|\, \varphi_\mu(x_1) \cdots \psi_\nu(x_n) \,|\, \Omega \rangle = i^F (-1)^J \langle \Omega \,|\, \psi_\nu(-x_n) \cdots \varphi_\mu(-x_1) \,|\, \Omega \rangle$$

  holds everywhere.

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  holds everywhere.

- In one sentence, CPT is always a symmetry of quantum field theory in flat spacetime.

A FUNDAMENTAL EXAMPLE

I. Overview

- The CPT symmetry of quantum field theory (in flat spacetime)
- The key point of the proof of the CPT theorem

$$\Lambda^0{}_0 \geqslant 1 \quad \det\Lambda = 1$$

$$\Lambda^0{}_0 \leqslant -1 \quad \det\Lambda = 1$$

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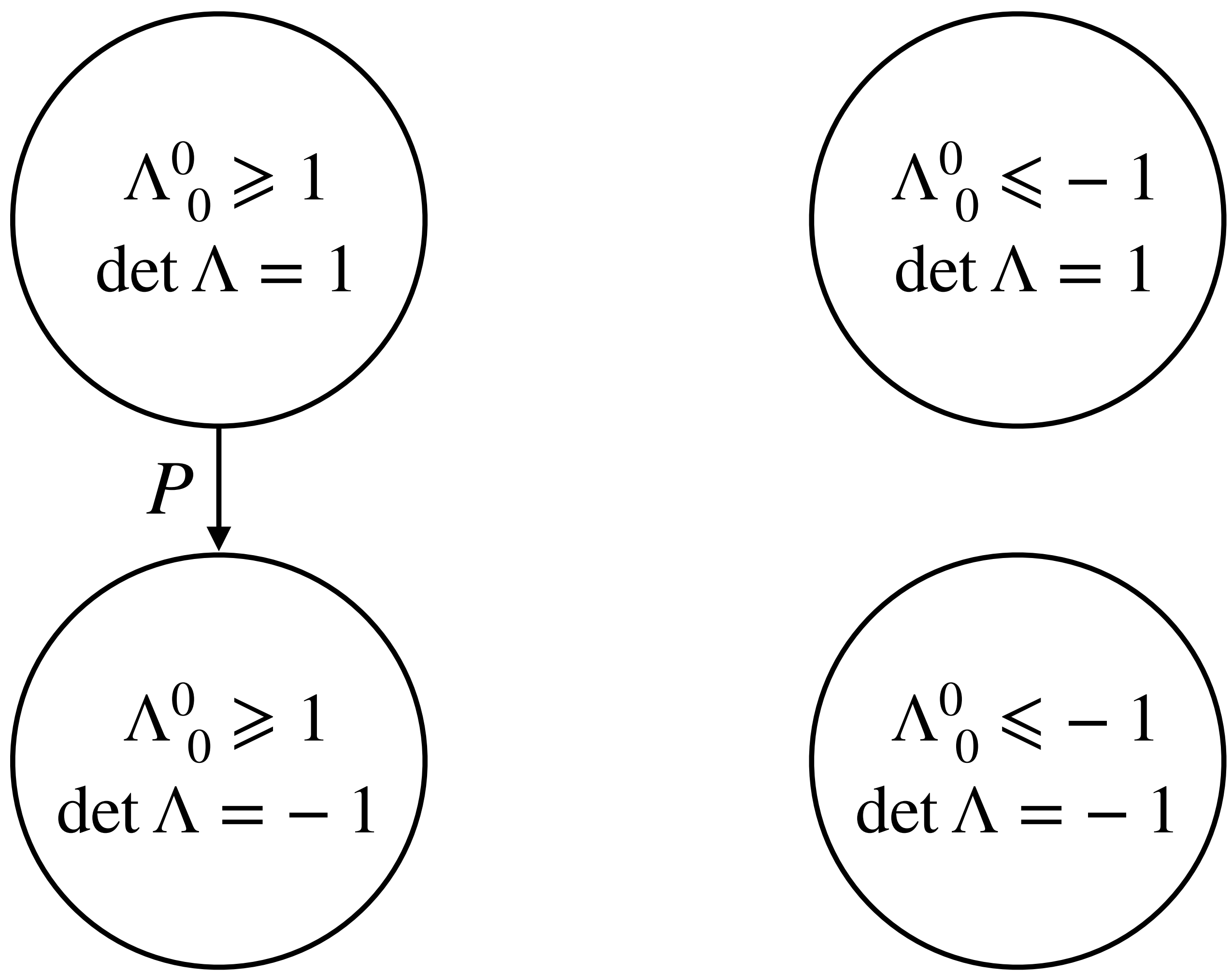

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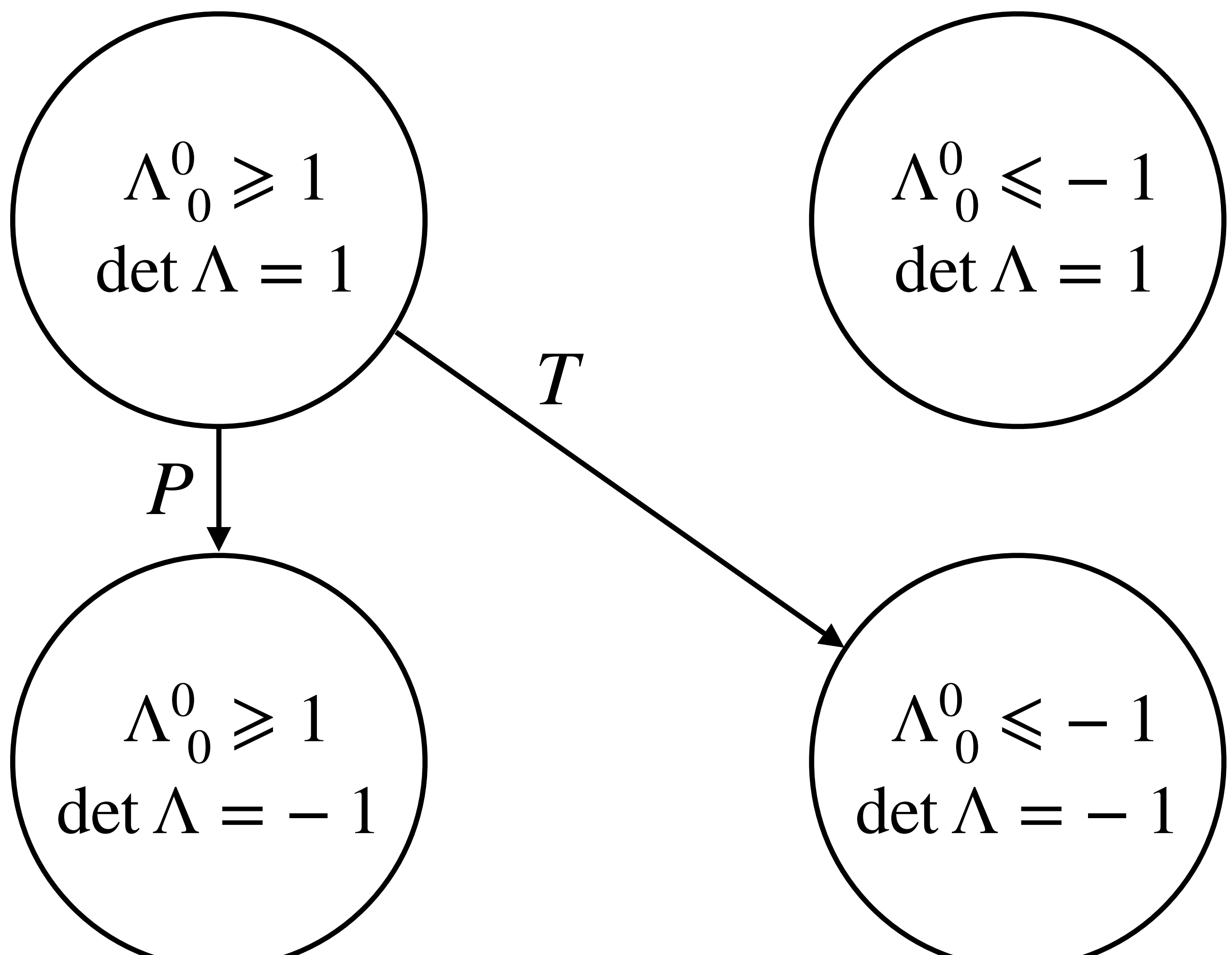

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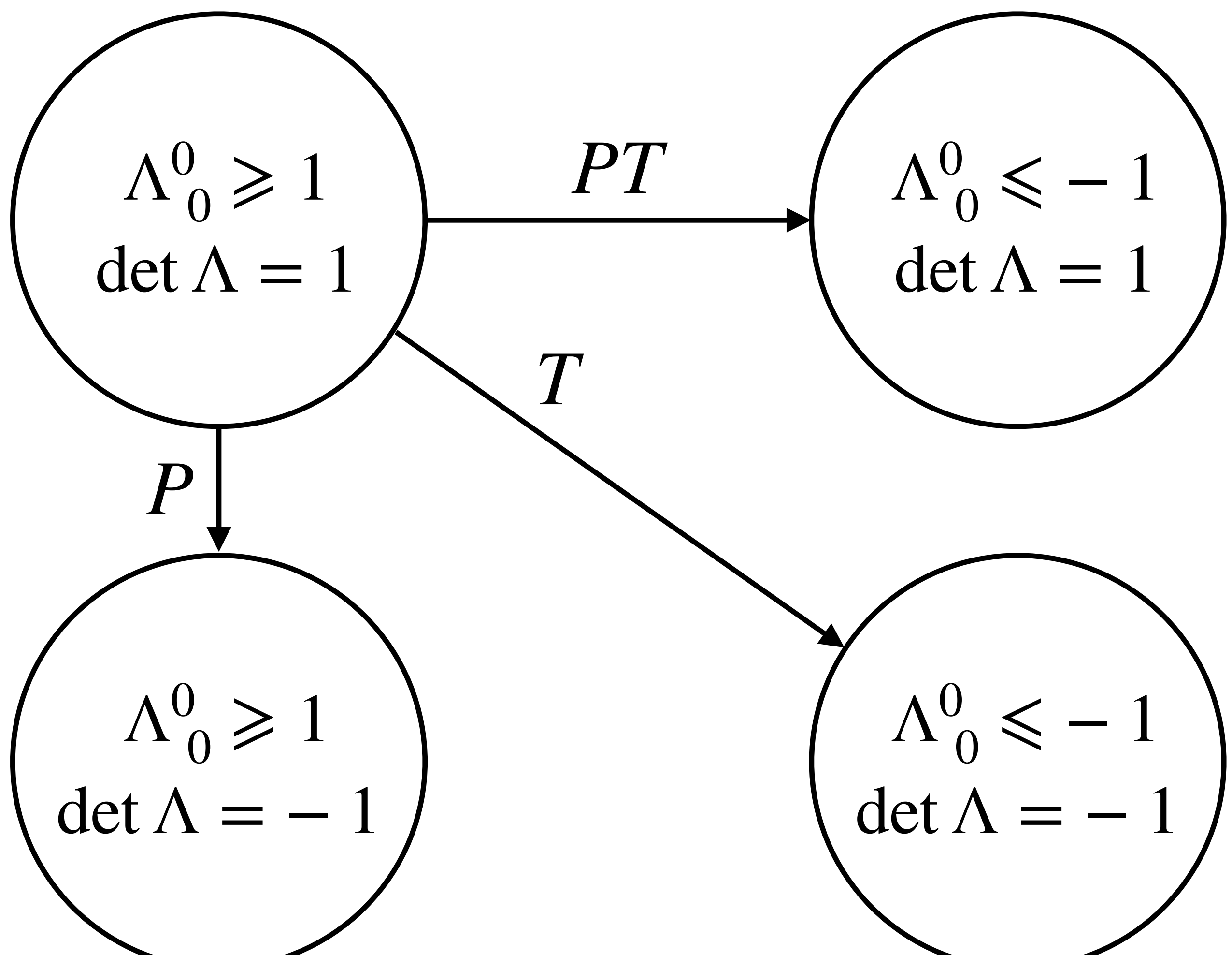

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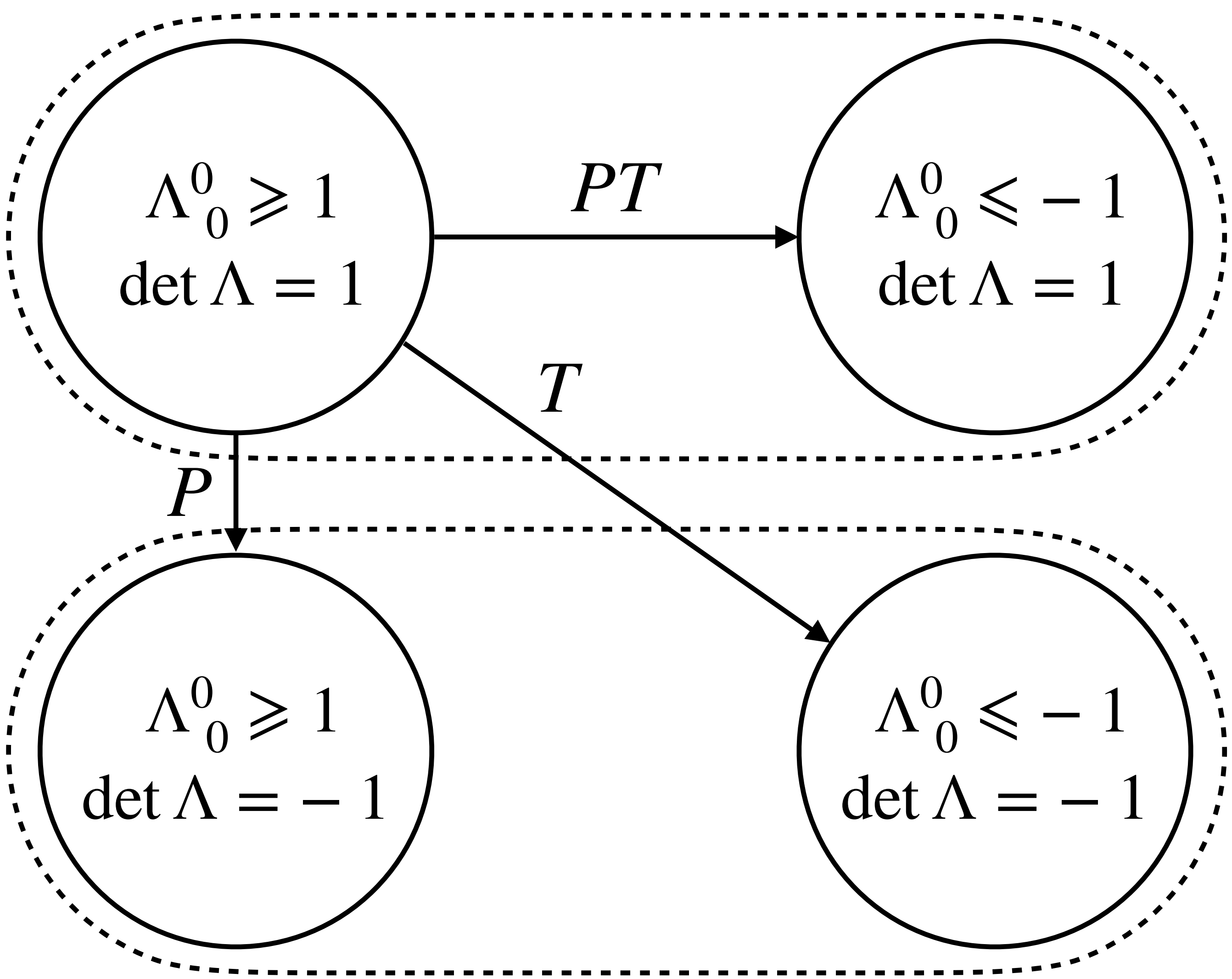

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- The CPT symmetry of quantum field theory (in flat spacetime)
- The key point of the proof of the CPT theorem: in complex Lorentz group, the PT transformation is in the same connected component with the identity element.

A FUNDAMENTAL EXAMPLE

I. Overview

- The CPT symmetry of quantum field theory (in flat spacetime)
- The key point of the proof of the CPT theorem: in complex Lorentz group, the PT transformation is in the same connected component with the identity element.
- However, in D-dimensional spacetime, one needs to replace the CPT transformation with the CRT transformation.
- R transformation: reflection of one space spatial coordinate.

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- **“The Euclidean path integrals are an effective way to compute the vacuum state (vacuum wave function) Ω of a quantum field theory. ”**

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- Path integral (quantum mechanics): how to calculate the transition amplitude?

- Wave function $\Psi(\mathbf{q}, t) = \langle \mathbf{q} \,|\, \Psi(t) \rangle_S = \langle \mathbf{q}, t \,|\, \Psi \rangle_H$, where $|\,\Psi(t)\rangle_S$ is the state vector in Schrödinger representation, $|\,\Psi\rangle_H$ is the state in Heisenberg representation.

- One wants to calculate the transition amplitude ${}_H\langle \Psi_2 \,|\, \Psi_1 \rangle_H$ with the knowledge of the initial state $\Psi_1(\mathbf{q}, t_i)$ and the final state $\Psi_2(\mathbf{q}, t_f)$.

A FUNDAMENTAL EXAMPLE

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- One wants to calculate the transition amplitude ${}_H\langle \Psi_2 | \Psi_1 \rangle_H$ with the knowledge of the initial state $\Psi_1(\mathbf{q}, t_i)$ and the final state $\Psi_2(\mathbf{q}, t_f)$.

- Because $\{ | \mathbf{q}, t \rangle \}$ is a complete base for every t, one has

$$
\begin{aligned}
{}_H\langle \Psi_2 | \Psi_1 \rangle_H &= \int d\mathbf{q}_f d\mathbf{q}_i \; {}_H\langle \Psi_2 | \mathbf{q}_f, t_f \rangle \langle \mathbf{q}_f, t_f | \mathbf{q}_i, t_i \rangle \langle \mathbf{q}_i, t_i | \Psi_1 \rangle_H \\
&= \int d\mathbf{q}_f d\mathbf{q}_i \Psi_2(\mathbf{q}_f, t_f)^* \langle \mathbf{q}_f, t_f | \mathbf{q}_i, t_i \rangle \Psi_1(\mathbf{q}_i, t_i)
\end{aligned}
$$

- The path integral tells us how to calculate the integral kernel (propagator)

$$
\langle \mathbf{q}_f, t_f | \mathbf{q}_i, t_i \rangle = \int_{\mathbf{q}(t_i)=\mathbf{q}_i}^{\mathbf{q}(t_f)=\mathbf{q}_f} [d\mathbf{q}(t)] \; \exp\left[\frac{i}{\hbar} \int_{t_i}^{t_f} dt \; L(\dot{\mathbf{q}}, \mathbf{q}) \right]
$$

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- Now let us “remove” ${}_H\langle\Psi_2|$ in the formula, then we have

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$$|\Psi_1\rangle_H = \int d\mathbf{q}_f d\mathbf{q}_i\ |\mathbf{q}_f, t_f\rangle\langle\mathbf{q}_f, t_f|\mathbf{q}_i, t_i\rangle\langle\mathbf{q}_i, t_i|\Psi_1\rangle_H$$

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$$= \int d\mathbf{q}_f d\mathbf{q}_i \; |\mathbf{q}_f, t_f\rangle\langle\mathbf{q}_f, t_f|\mathbf{q}_i, t_i\rangle\Psi_1(\mathbf{q}_i, t_i)$$

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&= \int d\mathbf{q}_f d\mathbf{q}_i \; |\mathbf{q}_f, t_f\rangle\langle\mathbf{q}_f, t_f|\,\mathbf{q}_i, t_i\rangle\Psi_1(\mathbf{q}_i, t_i) \\
&= \int d\mathbf{q}_f d\mathbf{q}_i \int_{\mathbf{q}(t_i)=\mathbf{q}_i}^{\mathbf{q}(t_f)=\mathbf{q}_f} [d\mathbf{q}(t)] \;\exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f} dt\; L(\dot{\mathbf{q}}, \mathbf{q})\right] \Psi_1(\mathbf{q}_i, t_i)\,|\,\mathbf{q}_f, t_f\rangle
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- And the wave function of the state $|\Psi_1\rangle_H$ is given by

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$$\Psi_1(\mathbf{q}, t) = \langle \mathbf{q}, t \,|\, \Psi_1\rangle_H = \int d\mathbf{q}_f d\mathbf{q}_i \ \langle \mathbf{q}, t \,|\, \mathbf{q}_f, t\rangle\langle \mathbf{q}_f, t \,|\, \mathbf{q}_i, t_i\rangle \Psi_1(\mathbf{q}_i, t_i)$$

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$$= \int d\mathbf{q}_i \Psi_1(\mathbf{q}_i, t_i) \int_{\mathbf{q}(t_i)=\mathbf{q}_i}^{\mathbf{q}(t)=\mathbf{q}} [d\mathbf{q}(t)]\ \exp\left[\frac{i}{\hbar}\int_{t_i}^{t} dt\ L(\dot{\mathbf{q}}, \mathbf{q})\right]$$

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- And the wave function of the state $|\Psi_1\rangle_H$ is given by

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- So we can also formally define $|\Psi_1\rangle_H$ by an path integral:

$$|\Psi_1\rangle_H = \int d\mathbf{q}_i \Psi_1(\mathbf{q}_i, t_i) \int_{\mathbf{q}(t_i)=\mathbf{q}_i} [d\mathbf{q}(t)] \ e^{iS/\hbar}$$

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- From quantum mechanics to quantum field theory

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$$\{\mathbf{q}(t_i), t_i\} \to \{\varphi(x), \Sigma_i\}$$

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$${}_H\langle\Psi_2|\Psi_1\rangle_H = \int \left(\mathscr{D}\varphi_f|_{\Sigma_f}\right)\left(\mathscr{D}\varphi_i|_{\Sigma_i}\right)\ {}_H\langle\Psi_2|\varphi_f\rangle\langle\varphi_f|\varphi_i\rangle\langle\varphi_i|\Psi_1\rangle_H$$

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&= \int \left(\mathscr{D}\varphi_f\right)\left(\mathscr{D}\varphi_i\right)\ \Psi_2^*[\varphi_f]\Psi_1[\varphi_i]\int_{\varphi(\Sigma_i)=\varphi_i}^{\varphi(\Sigma_f)=\varphi_f}[\mathscr{D}\varphi]\exp\left[\frac{i}{\hbar}\int d^4x\mathscr{L}(\varphi,\partial_\mu\varphi)\right]
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$$\Psi[\varphi] = \int \left(\mathscr{D}\varphi_i\right)\ \Psi[\varphi_i]\int_{\varphi(\Sigma_i)=\varphi_i}^{\varphi(\Sigma)=\varphi}[\mathscr{D}\varphi]\exp\left[\frac{i}{\hbar}\int d^4x\mathscr{L}(\varphi, \partial_\mu\varphi)\right]$$

A FUNDAMENTAL EXAMPLE

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&= \int \left(\mathscr{D}\varphi_f \right) \left(\mathscr{D}\varphi_i \right) \; \Psi_2^*[\varphi_f] \Psi_1[\varphi_i] \int_{\varphi(\Sigma_i)=\varphi_i}^{\varphi(\Sigma_f)=\varphi_f} [\mathscr{D}\varphi] \exp\left[\frac{i}{\hbar} \int d^4x \mathscr{L}(\varphi, \partial_\mu \varphi) \right]
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$$|\Psi\rangle_H = \int \left(\mathscr{D}\varphi_i \right) \; \Psi[\varphi_i] \int_{\varphi(\Sigma_i)=\varphi_i} [\mathscr{D}\varphi] e^{iS/\hbar}$$

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- Vacuum to vacuum amplitude

A FUNDAMENTAL EXAMPLE

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$$\langle \varphi_f(t_f) \,|\, \varphi_i(t_i) \rangle = \sum_{n,m} \langle \varphi_f(t_f) \,|\, n \rangle \langle n \,|\, m \rangle \langle m \,|\, \varphi_i(t_i) \rangle$$

A FUNDAMENTAL EXAMPLE

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- Vacuum to vacuum amplitude

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$$= \sum_{n,m} \langle \varphi_f(0) | e^{-i\hat{H}t_f} | n \rangle \langle n | m \rangle \langle m | e^{i\hat{H}t_i} | \varphi_i(0) \rangle = \sum_{n,m} e^{-iE_n t_f + iE_m t_i} \langle \varphi_f(0) | n \rangle \langle n | m \rangle \langle m | \varphi_i(0) \rangle$$

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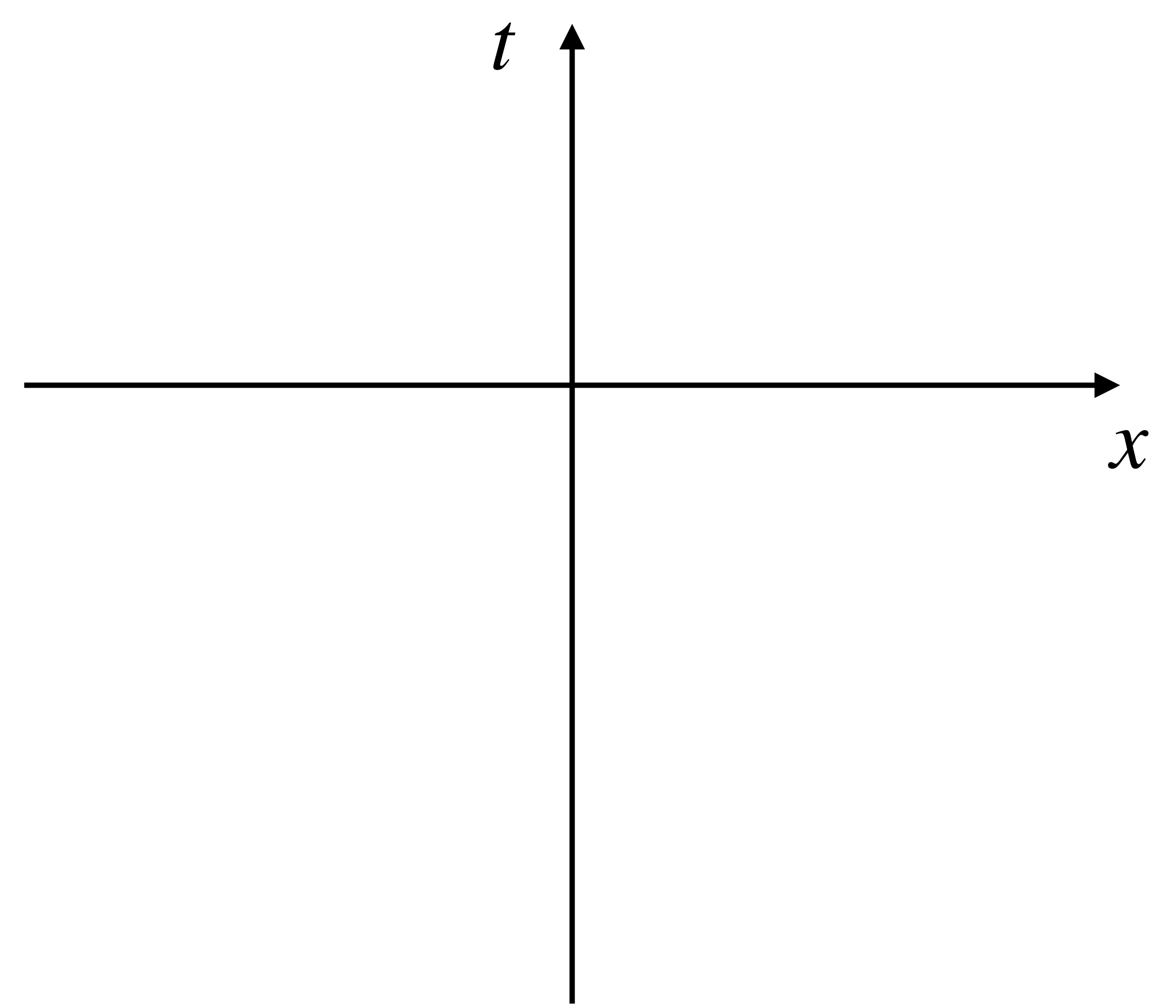
- So $\langle \varphi_f(0) | \varphi_i(-\infty) \rangle = \langle \varphi_f(0) | \Omega \rangle \langle \Omega | \varphi_i(0) \rangle$ for any $\langle \varphi_f(0) |$, which gives $| \varphi_i(-\infty) \rangle \propto | \Omega \rangle$.

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- This result tells us that the path integral on the half-space $t < 0$ as a functional of the boundary values of the fields $\varphi(0)$ gives a way to compute the vacuum wave functional $\Omega[\varphi]$.

$$\Omega[\varphi] \propto \langle \varphi(0) \,|\, \varphi_i(-\infty) \rangle$$



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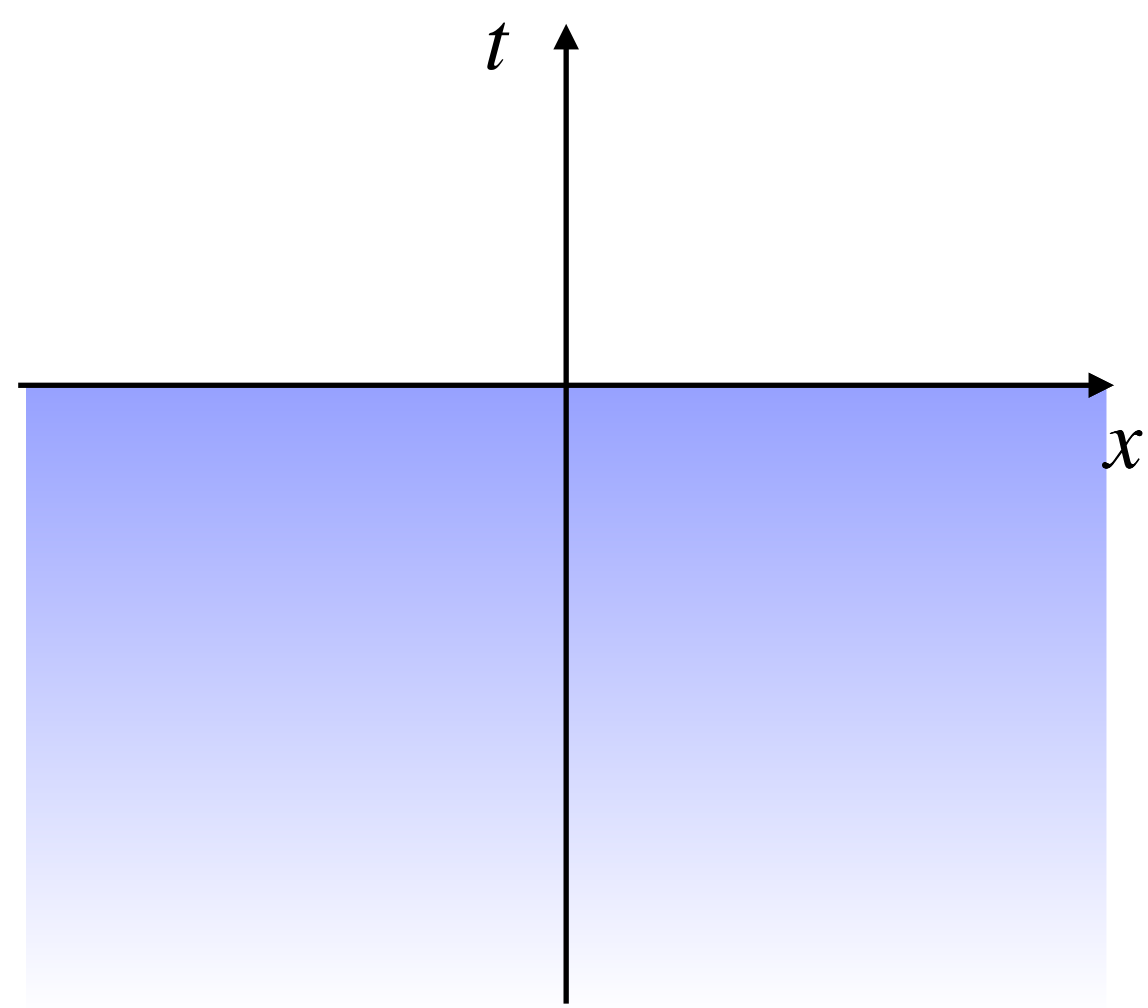

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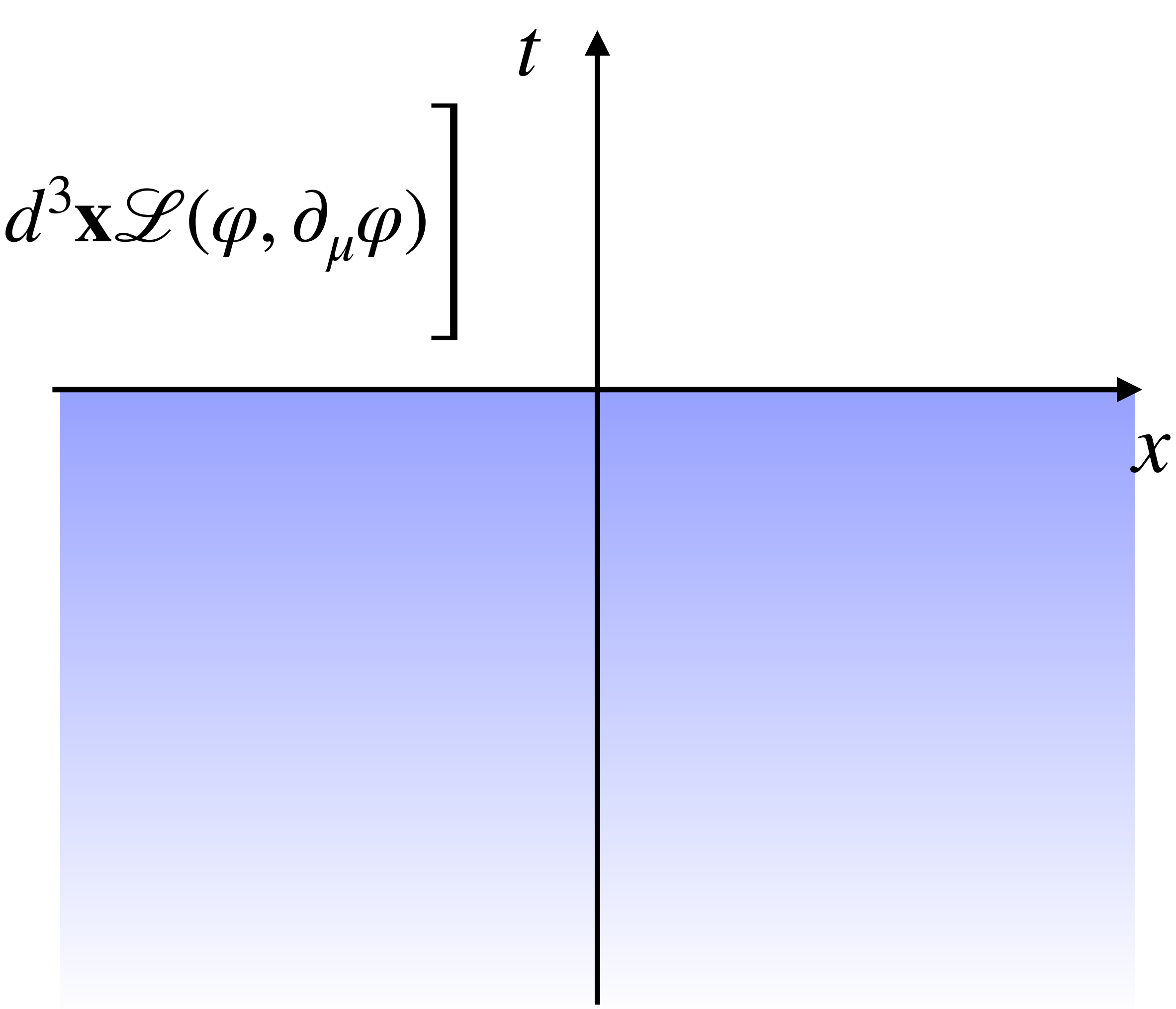
$$\Omega[\varphi] \propto \langle \varphi(0) \,|\, \varphi_i(-\infty) \rangle$$
$$= \int^{\varphi(0)} [\mathscr{D}\varphi] \exp\left[\frac{i}{\hbar} \int_{-\infty}^{0} dt \int d^3\mathbf{x} \mathscr{L}(\varphi, \partial_\mu \varphi) \right]$$



A FUNDAMENTAL EXAMPLE

II. Path integral approach

- From Minkowski metric to Euclidean metric: $t \to -i\tau$

Konrad Osterwalder
(1942/07/03-)

Robert Schrader
(1939/09/12-2015/11/29)

A FUNDAMENTAL EXAMPLE

II. Path integral approach

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$$\varphi_E(\mathbf{x}, t) \equiv \varphi_M(\mathbf{x}, it)$$
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A FUNDAMENTAL EXAMPLE

II. Path integral approach

- From Minkowski metric to Euclidean metric: $t \to -i\tau$

$$\varphi_E(\mathbf{x}, t) \equiv \varphi_M(\mathbf{x}, it)$$
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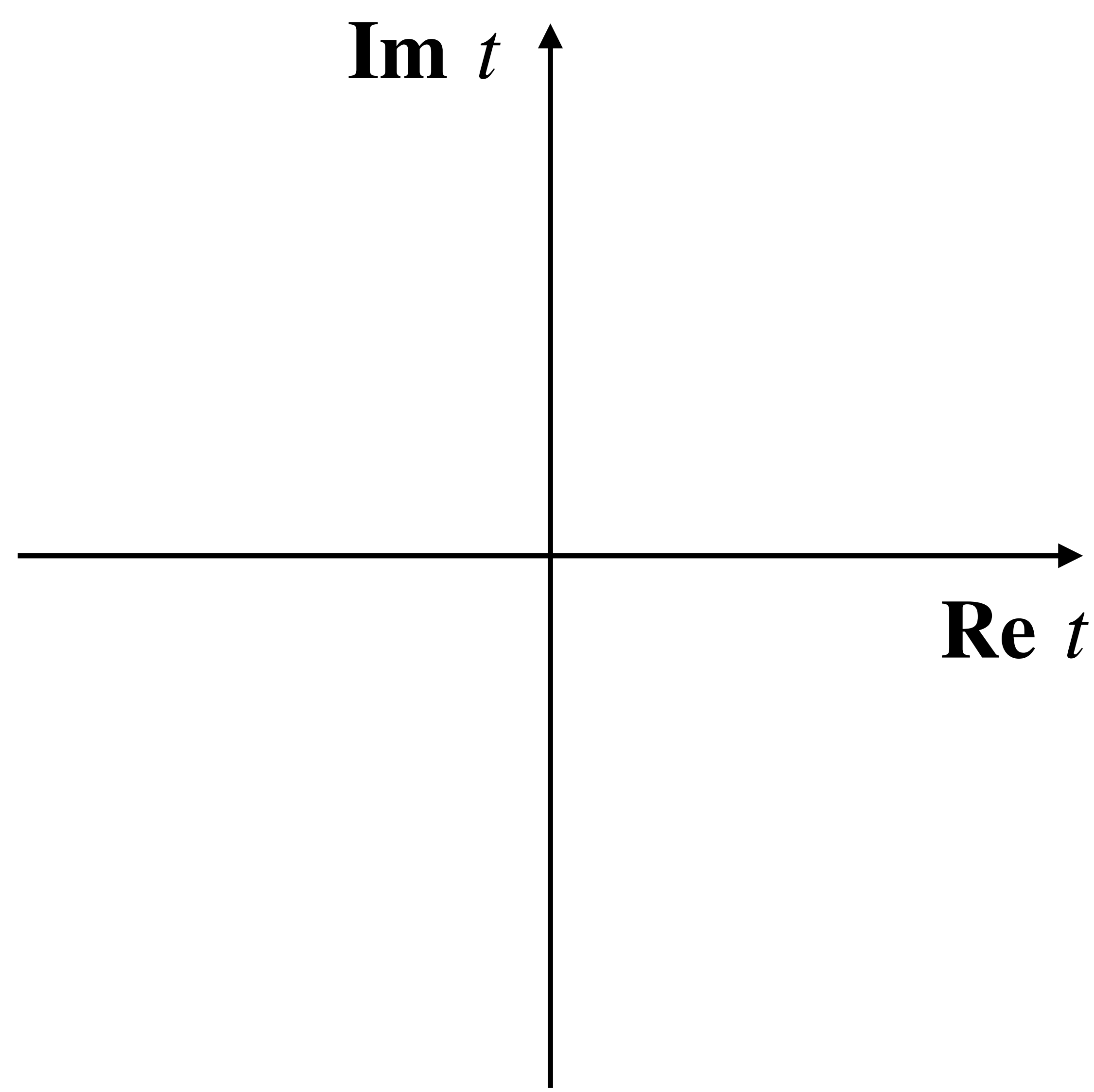

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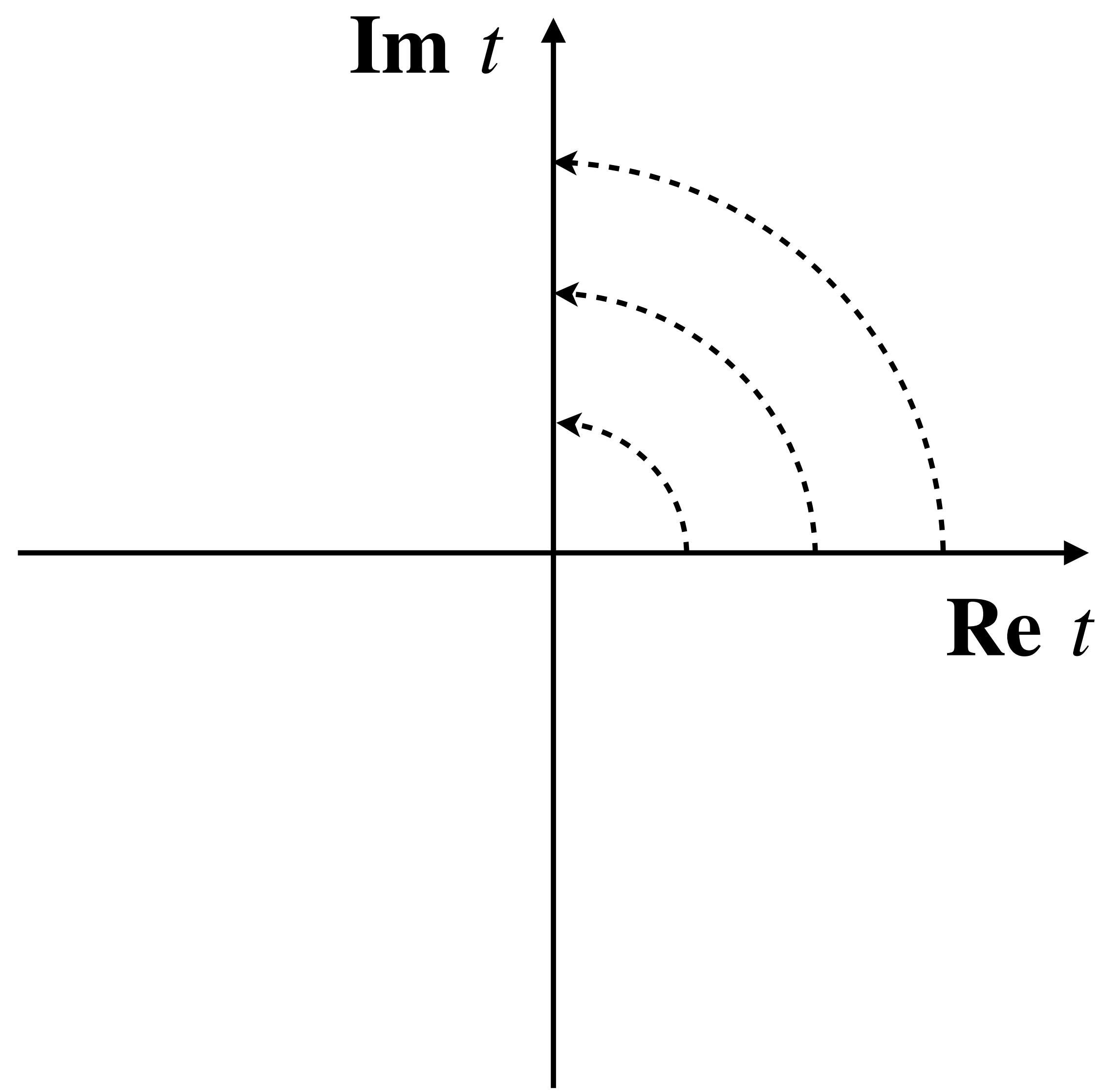

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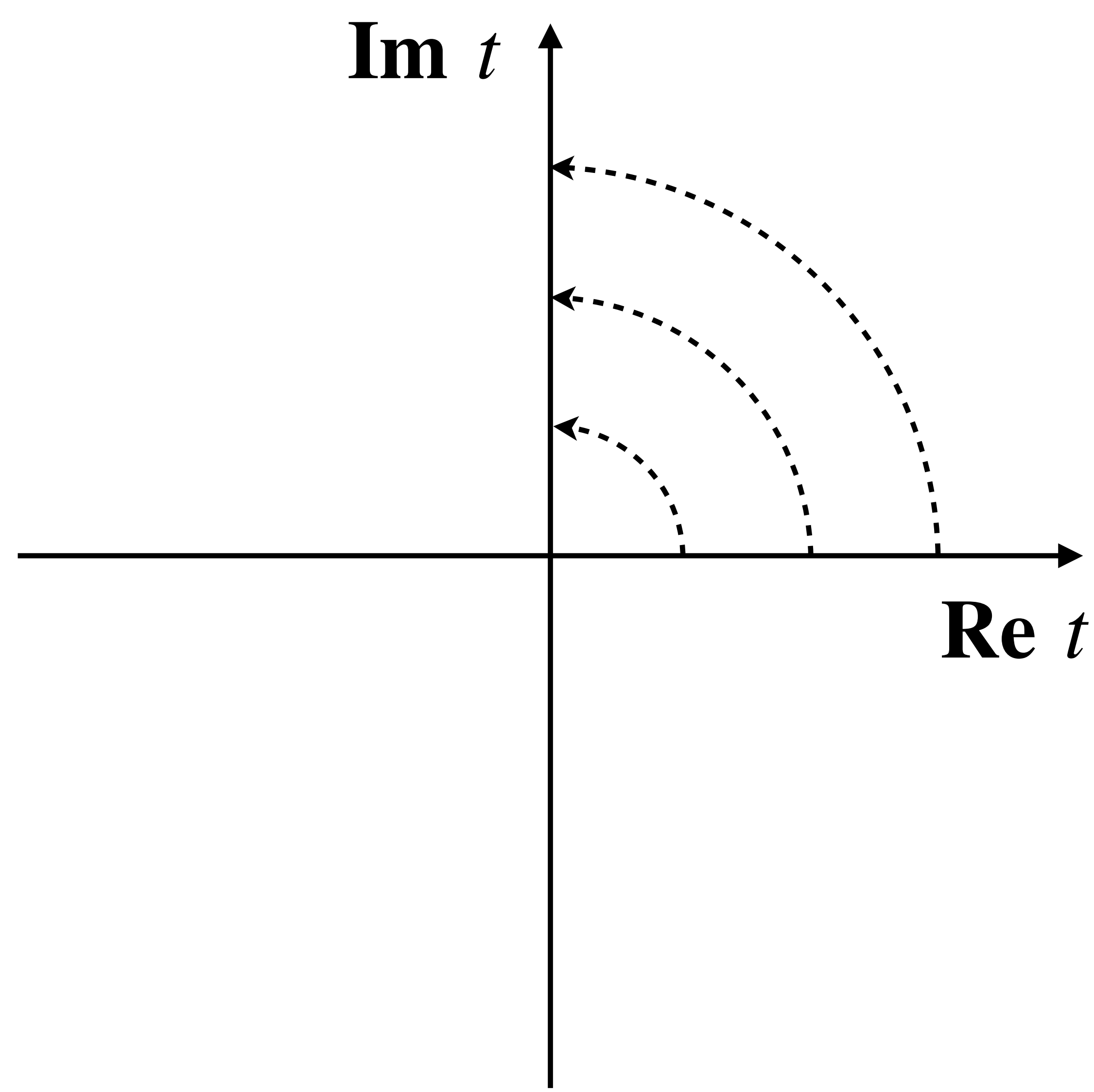
$$e^{iS/\hbar} \to e^{-S/\hbar}$$



A FUNDAMENTAL EXAMPLE

II. Path integral approach

- From Minkowski metric to Euclidean metric: $t \to -i\tau$

- The vacuum wave functional can be calculated with Euclidean path integral.

- If the Hilbert space $\mathscr{H}$ of a quantum field theory can be factorized as $\mathscr{H} = \mathscr{H}_\ell \otimes \mathscr{H}_r$, where $\mathscr{H}_\ell$ and $\mathscr{H}_r$ are Hilbert spaces of degrees of freedom located at left-wedge and right-wedge, respectively, what we want to calculate is the partial trace over $\mathscr{H}_\ell$ of the density matrix $|\Omega\rangle\langle\Omega|$.

$$\rho_r = \mathbf{Tr}_{\mathscr{H}_\ell}|\Omega\rangle\langle\Omega| = ?$$

A FUNDAMENTAL EXAMPLE

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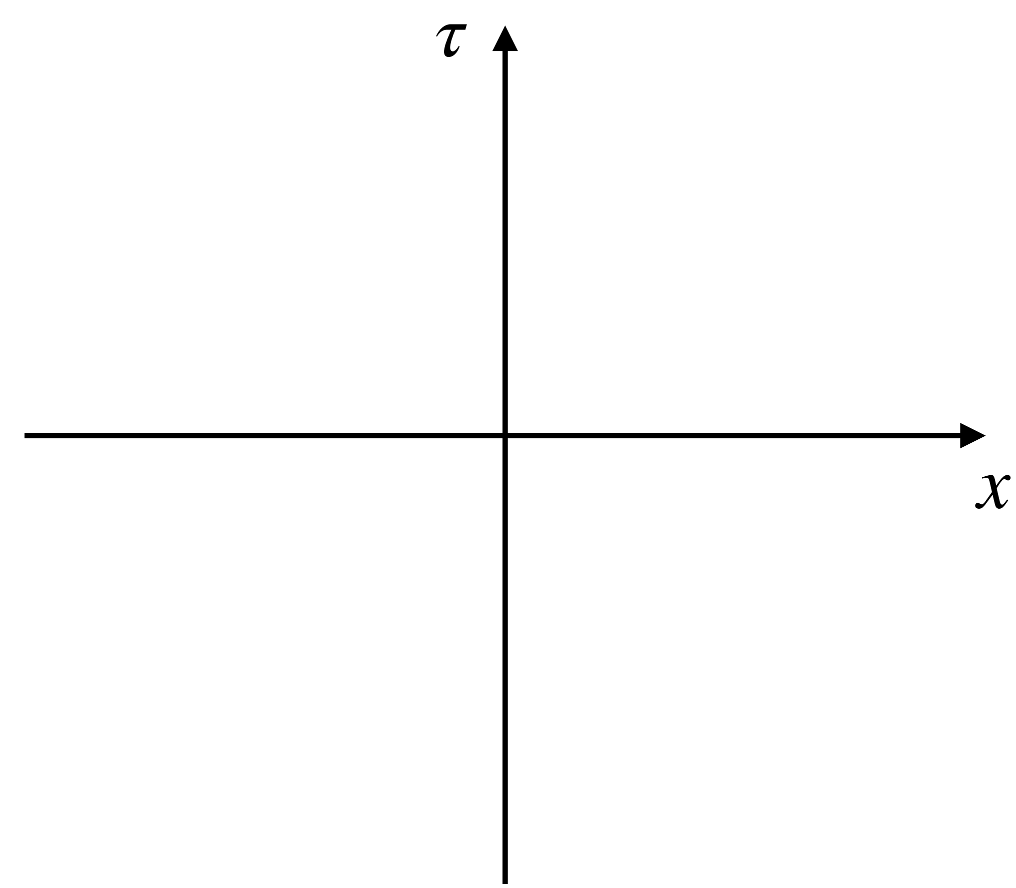
- The boundary condition φ of the quantum fields at $\tau = 0$ can be separated to the boundary conditions on the left half-space φ_ℓ and the boundary conditions on the left half-space φ_r.



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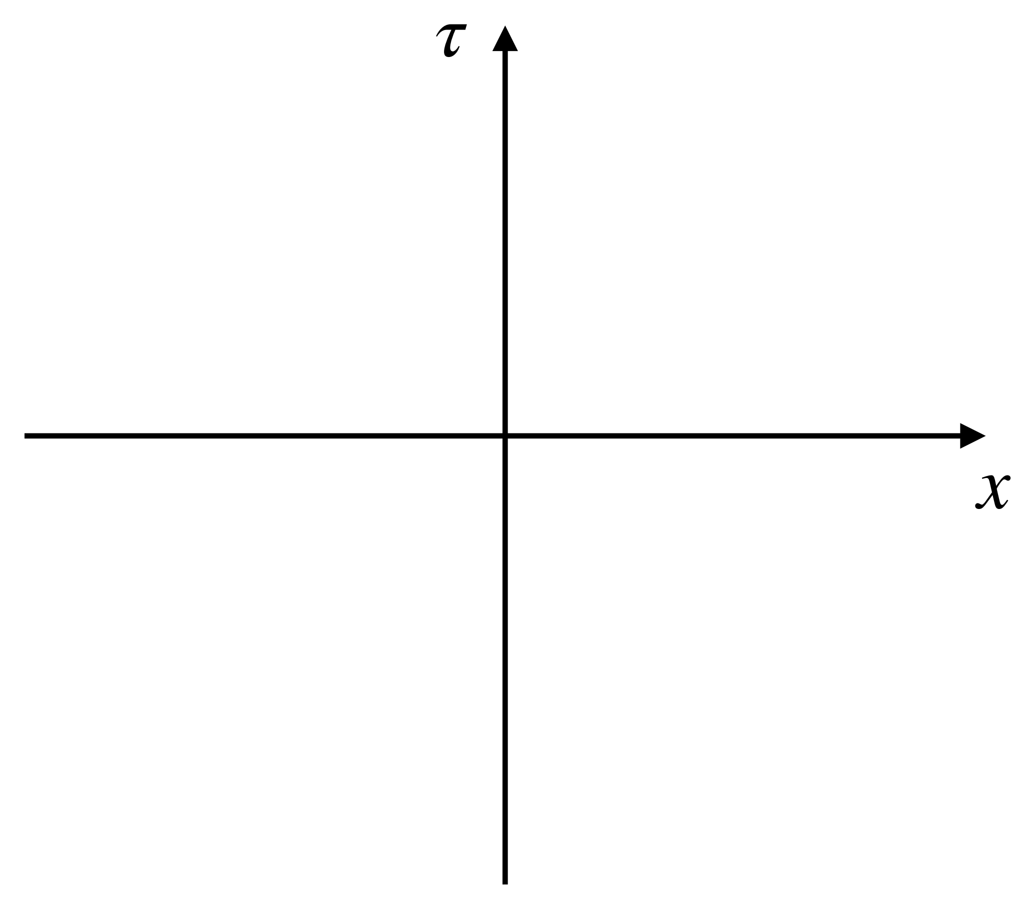

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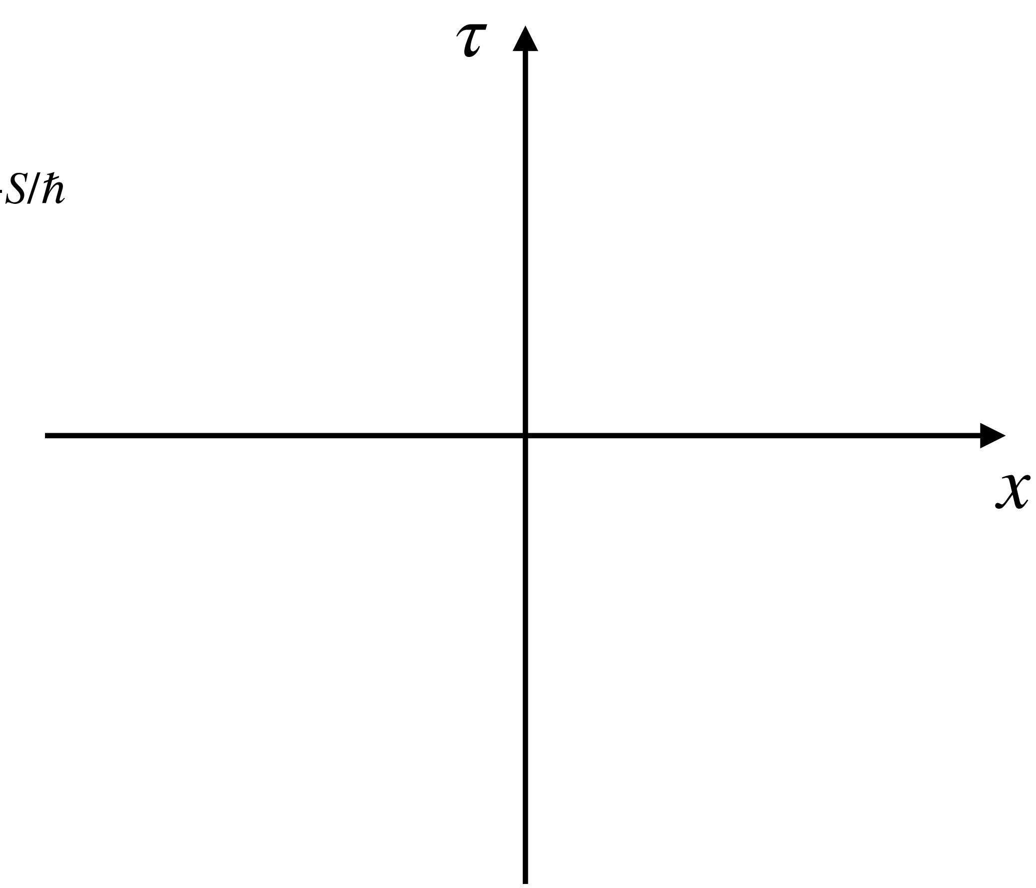

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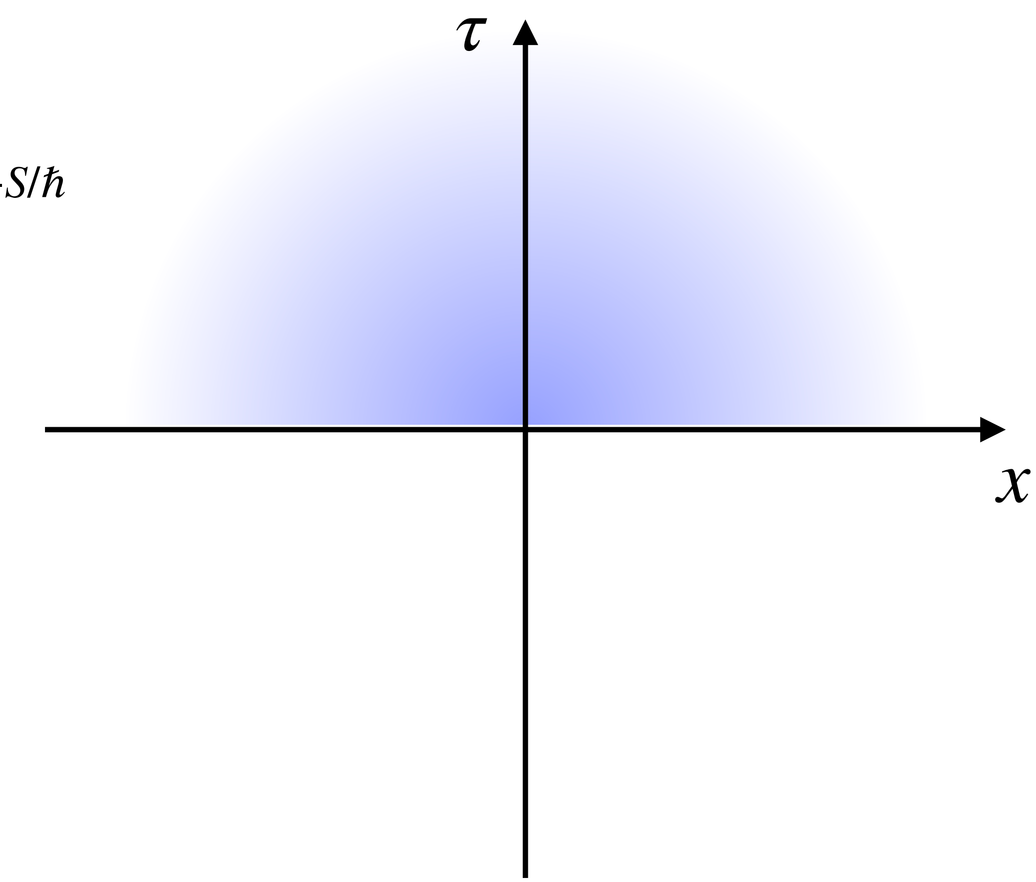

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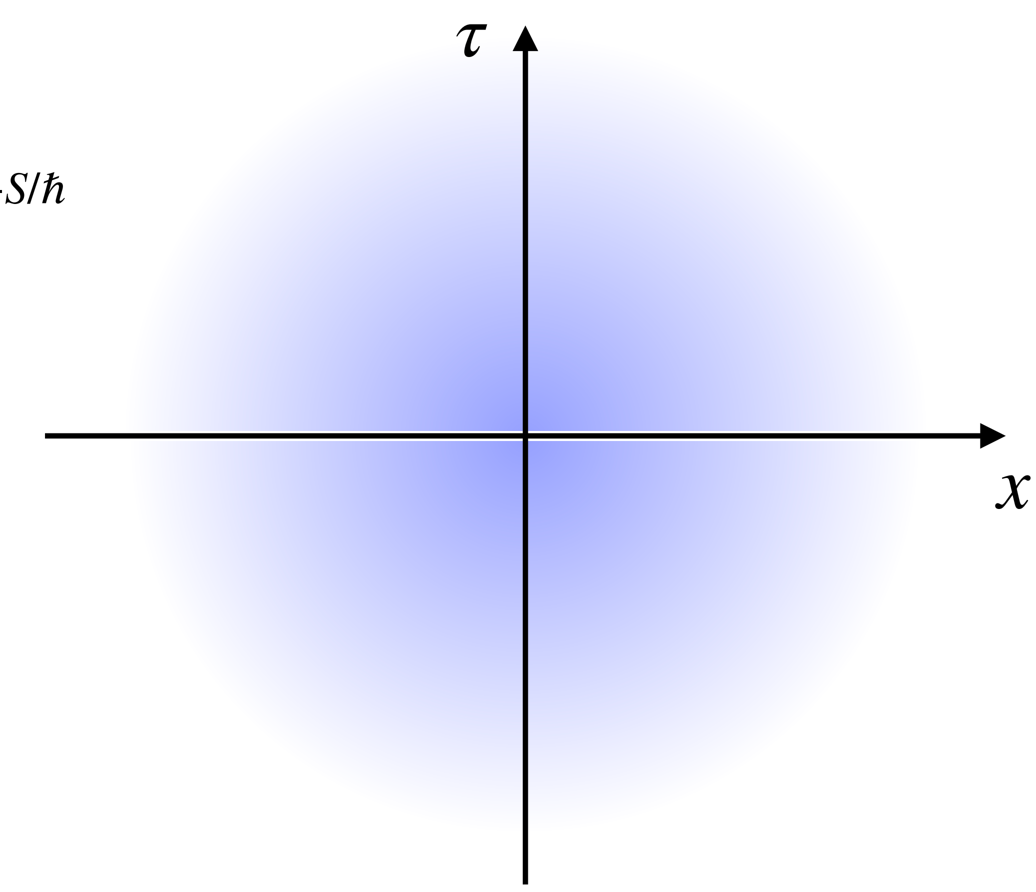

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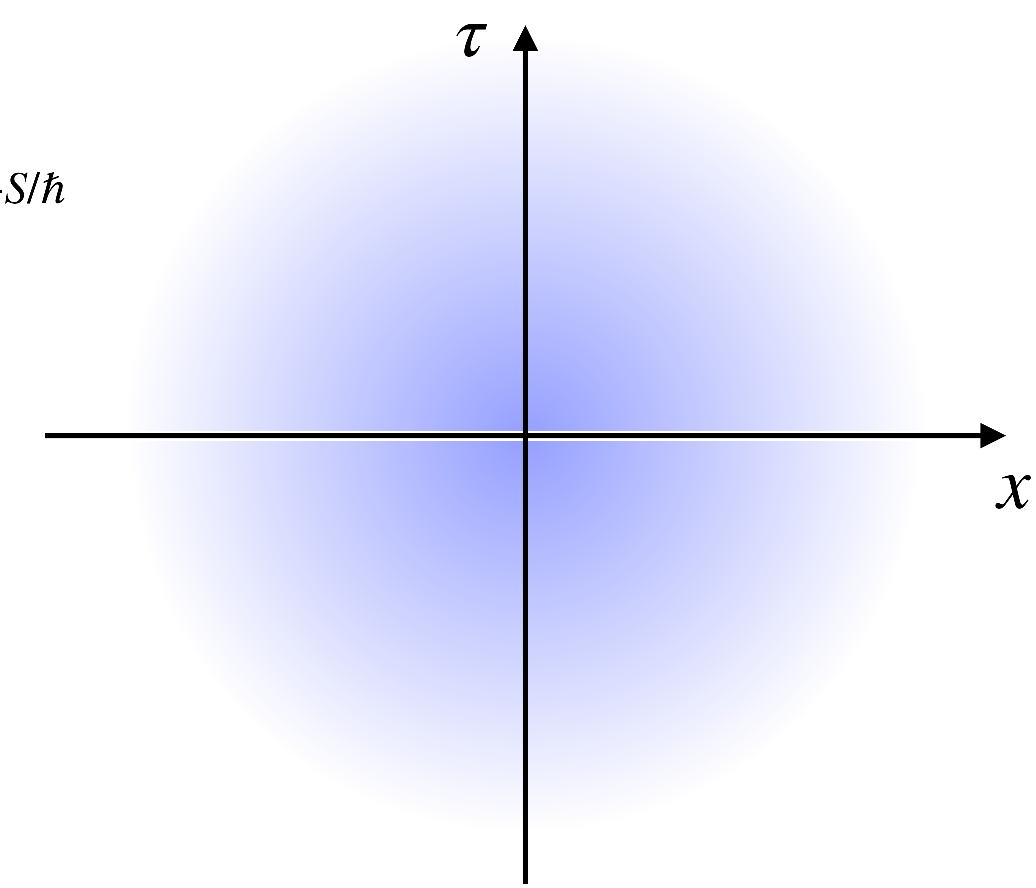

$$\therefore\ \hat{\rho}_r[\varphi_r, \varphi'_r] = \int [\mathscr{D}\varphi_\ell]\,|\,\Omega(\varphi_\ell, \varphi'_r)\rangle\langle\Omega(\varphi_\ell, \varphi_r)\,|$$

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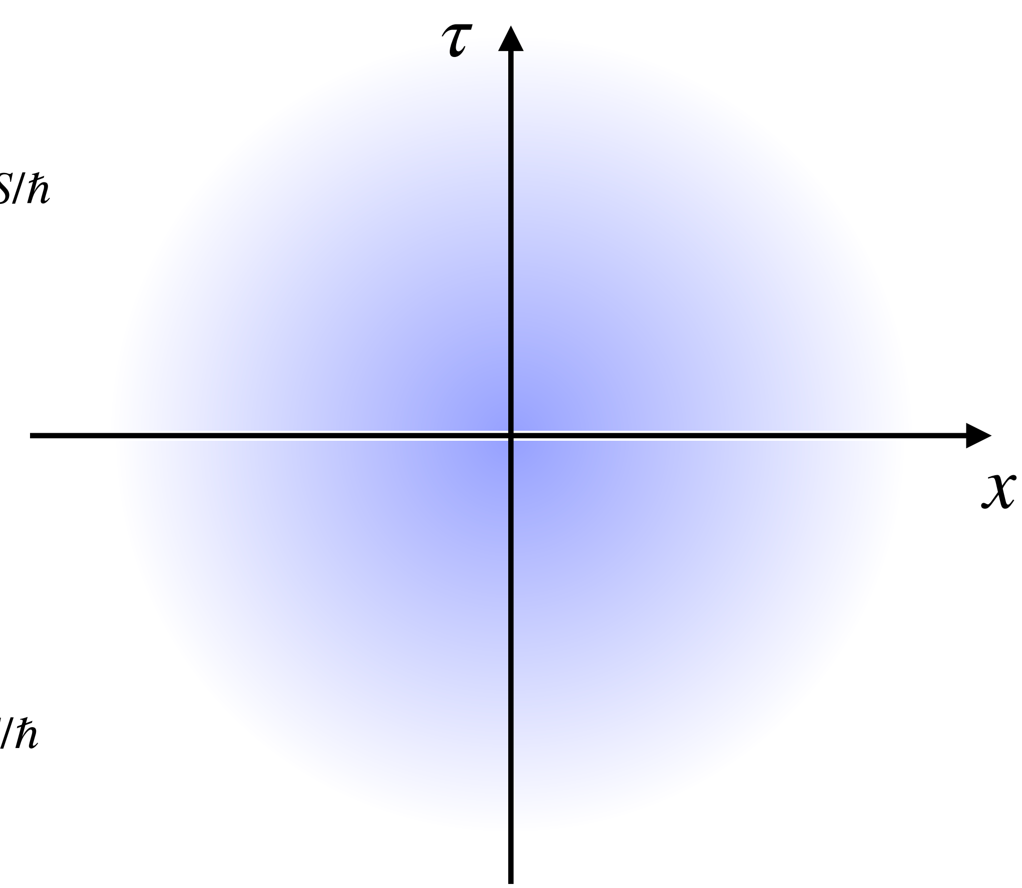

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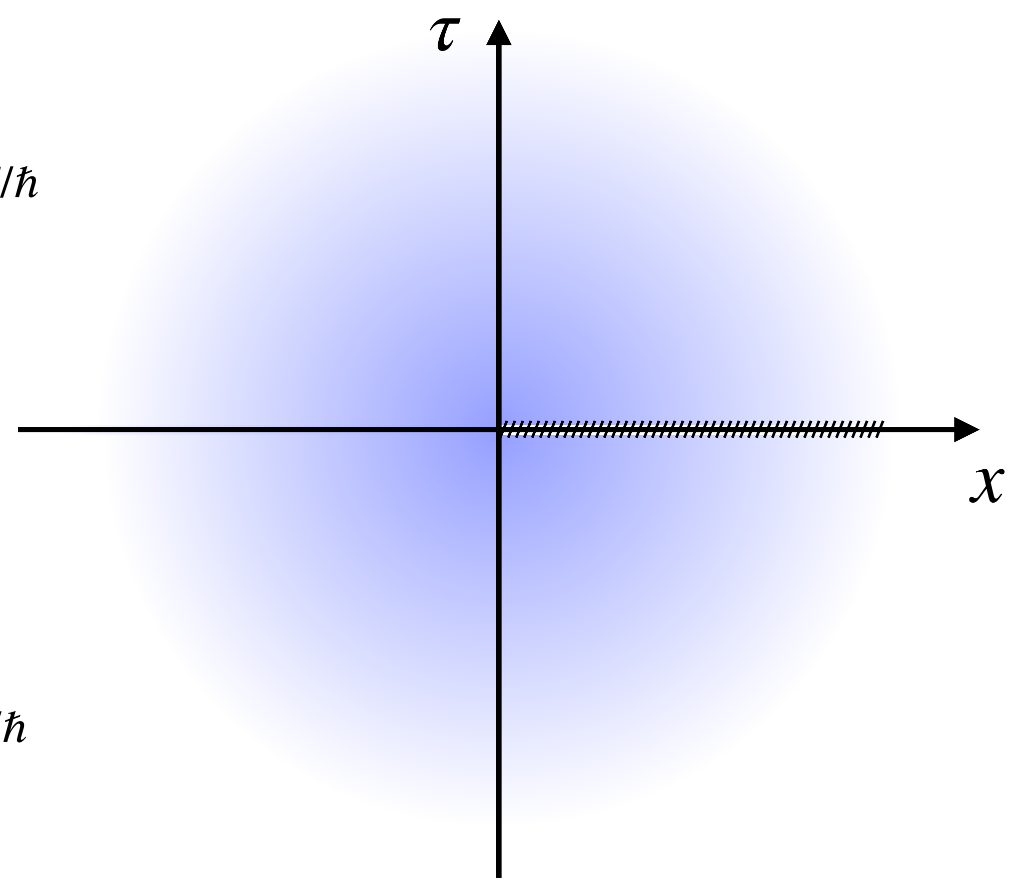

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II. Path integral approach

- The boundary condition φ of the quantum fields at $\tau = 0$ can be separated to the boundary conditions on the left half-space φ_ℓ and the boundary conditions on the left half-space φ_r.

- So the gluing gives a spacetime $W_{2\pi}$ (wedge-2π), a copy of Euclidean space except that it has been "cut" along the half-hyperplane $\tau = 0,\, x > 0$.

- In this path integral, the φ_r and φ_r' are the boundary values below and above the cut.

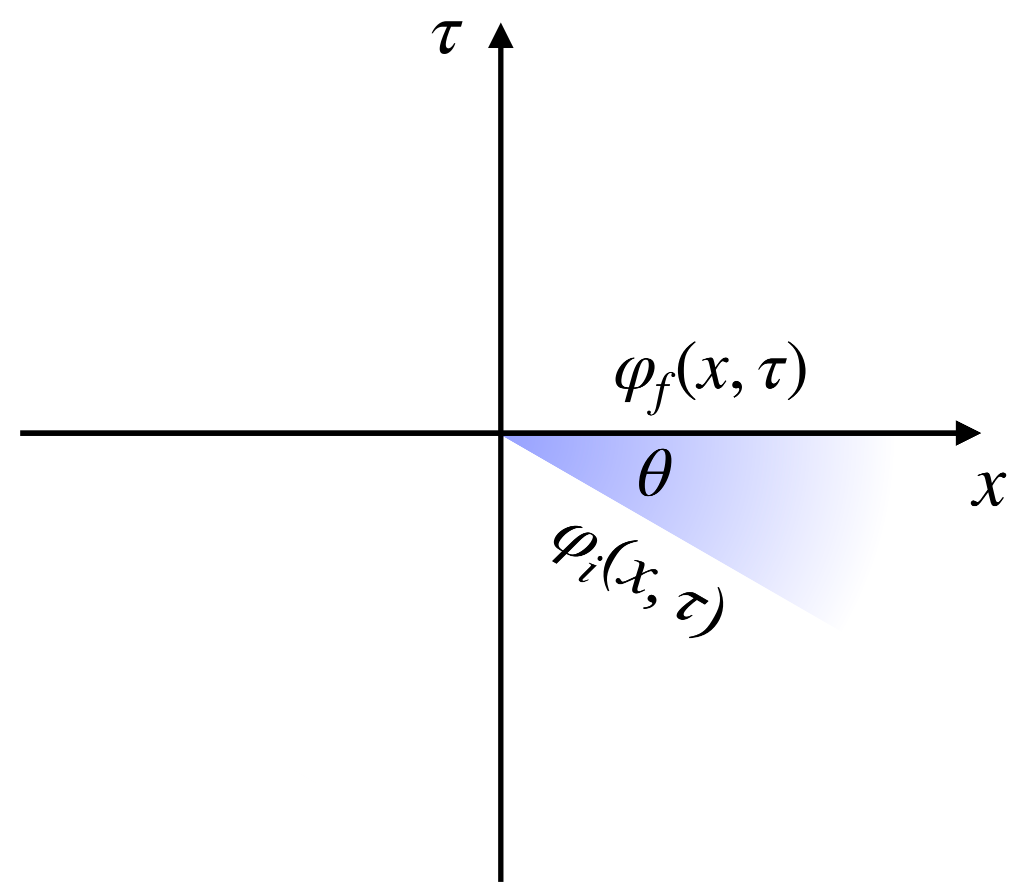
- How to calculate the path integral?

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- Considering the wedge W_θ of opening angle θ.
- Euclidean rotation

$$R_\theta \begin{pmatrix} \tau \\ x \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tau \\ x \end{pmatrix}$$



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- The operator translates the initial value surface to the final surface.

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$$\langle \varphi_f(x, \tau_f) \,|\, \varphi_i(x, \tau_i) \rangle = \int_{\varphi(\tau_i)=\varphi_i}^{\varphi(\tau_f)=\varphi_f} [\mathscr{D}\varphi] e^{-S/\hbar}$$

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$$\langle \varphi_f(x, \tau_f) \,|\, \varphi_i(x, \tau_i) \rangle = \langle \varphi_f(x,0) \,|\, U(\tau_f,0) U(0,\tau_i) \,|\, \varphi_i(x,0) \rangle = \langle \varphi_f(x,0) \,|\, \exp(-\hat{H}(\tau_f - \tau_i)) \,|\, \varphi_i(x,0) \rangle$$

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A FUNDAMENTAL EXAMPLE

II. Path integral approach

- Going back to Minkowski spacetime $\tau = it$:

$$R_\theta \begin{pmatrix} t \\ x \end{pmatrix} \to R_\theta \begin{pmatrix} \tau \\ x \end{pmatrix} = \begin{pmatrix} \tau\cos\theta + x\sin\theta \\ -\tau\sin\theta + x\cos\theta \end{pmatrix} = \begin{pmatrix} iR(\theta)t \\ R(\theta)x \end{pmatrix}$$

$$\therefore \quad R(\theta)t = -i\tau\cos\theta - ix\sin\theta = t\cos\theta - ix\sin\theta$$
$$R(\theta)x = -\tau\sin\theta + x\cos\theta = -it\sin\theta + x\sin\theta$$

$$\Rightarrow \quad R_\theta \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh(-i\theta) & \sinh(-i\theta) \\ \sinh(-i\theta) & \cosh(-i\theta) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

- So the wedge path integral W_θ in Euclidean space is a Lorentz boost of the $t - x$ plane by an imaginary boost parameter $-i\theta$.

A FUNDAMENTAL EXAMPLE

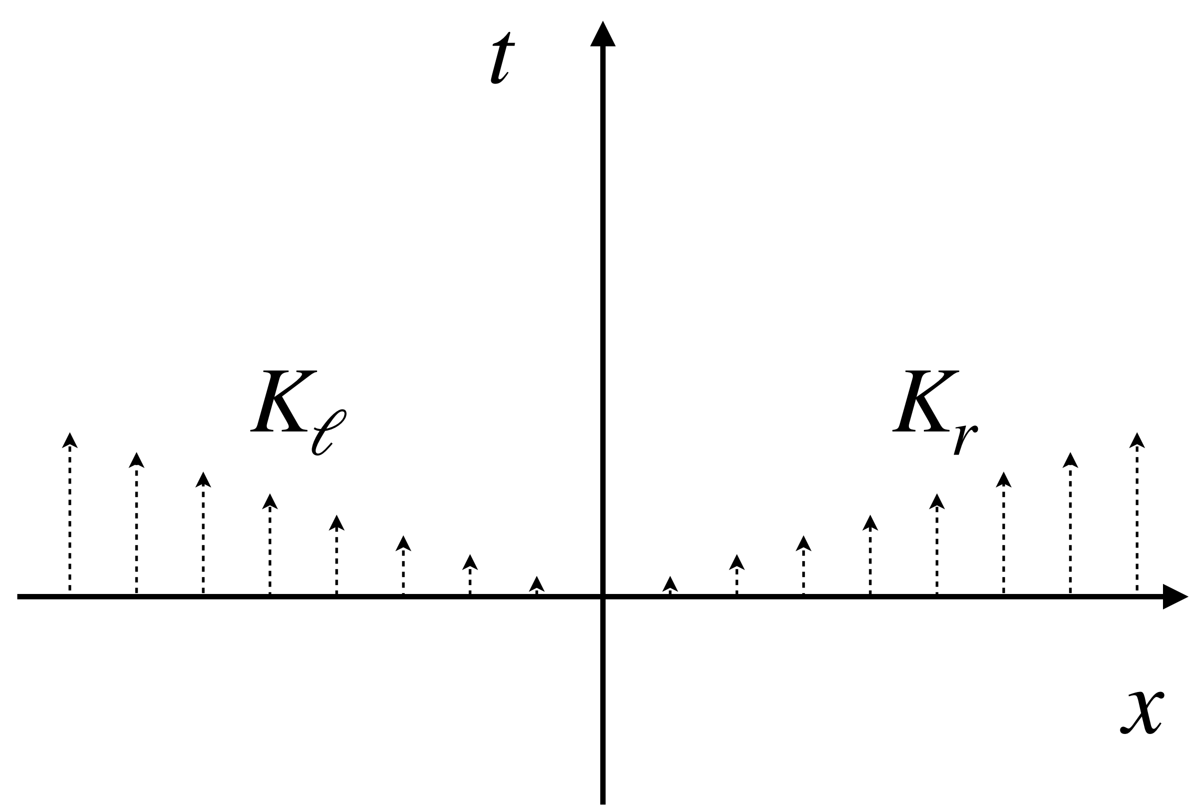
II. Path integral approach

- One can formally separate the boost generator to the left half-space part K_ℓ and the right half-space part K_r.

$$K_\ell \equiv \int_{-\infty}^{0} dx \int d^{D-2}\mathbf{y}\ (-x)T^{00} = -\int_{-\infty}^{0} dx \int d^{D-2}\mathbf{y}\ xT^{00}$$

$$K_r = \int_{0}^{+\infty} dx \int d^{D-2}\mathbf{y}\ xT^{00}$$

$$K = \int_{-\infty}^{\infty} dx d^{D-2}\mathbf{y}\ xT^{00} = K_r - K_\ell$$



A FUNDAMENTAL EXAMPLE

II. Path integral approach

- One can formally separate the boost generator to the left half-space part K_ℓ and the right half-space part K_r.

- Although $K = K_r - K_\ell$ is a well-defined operator, K_ℓ and K_r have well-defined matrix elements $\langle \chi | K_\ell | \psi \rangle$ and $\langle \chi | K_r | \psi \rangle$ between suitable Hilbert space states $|\chi\rangle$ and $|\psi\rangle$, if one tries to compute the norm of the state $K_\ell |\chi\rangle$ or $K_r |\chi\rangle$, one will find a universal UV-divergence near $x = 0$, independent of the choice of $|\chi\rangle$.

- This is related to the fact that the factorization $\mathscr{H} = \mathscr{H}_\ell \otimes \mathscr{H}_r$ is not really correct.

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- The unitary operator generated by the self-adjoint operator K with a real boost parameter η is $\exp(-i\eta K)$.

- When $\eta = -i\theta$, the operator becomes $\exp(-\theta K)$.

- So in real time language, the path integral on the wedge W_θ constructs the operator $\exp(-\theta K_r)$.

- To get the reduced density matrix ρ_r, we need to set $\theta = 2\pi$:

$$\rho_r = \exp(-2\pi K_r)$$

A FUNDAMENTAL EXAMPLE

II. Path integral approach

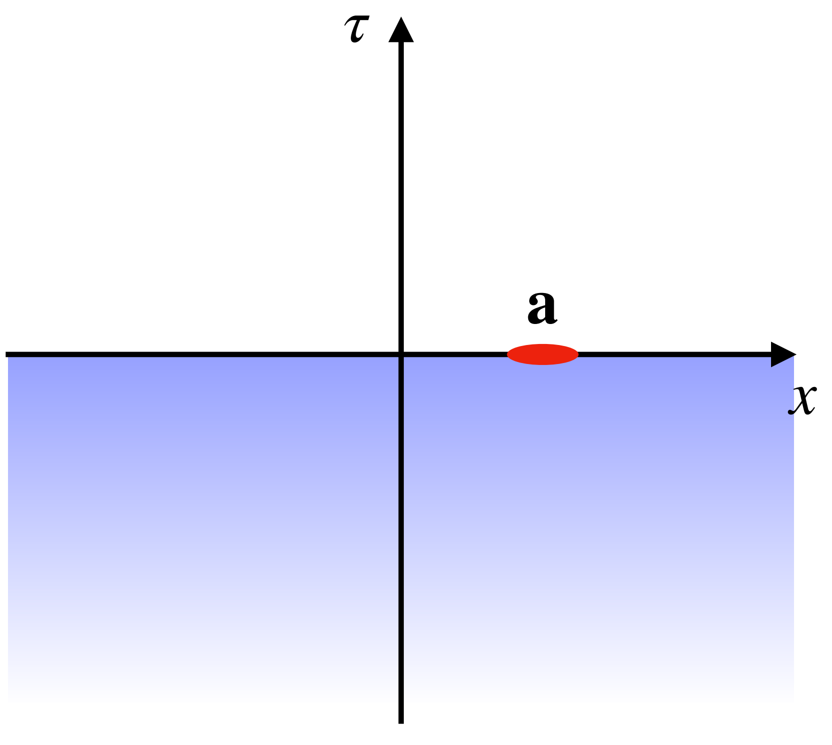
- With the assumption that $\mathscr{H} = \mathscr{H}_\ell \otimes \mathscr{H}_r$ (which is not correct), we have (because $[K_\ell, K_r] = 0$)

$$\Delta_\Omega = \rho_r \otimes \rho_\ell^{-1} = \exp(-2\pi K_r)\exp(2\pi K_\ell) = \exp(-2\pi K)$$

A FUNDAMENTAL EXAMPLE

II. Path integral approach

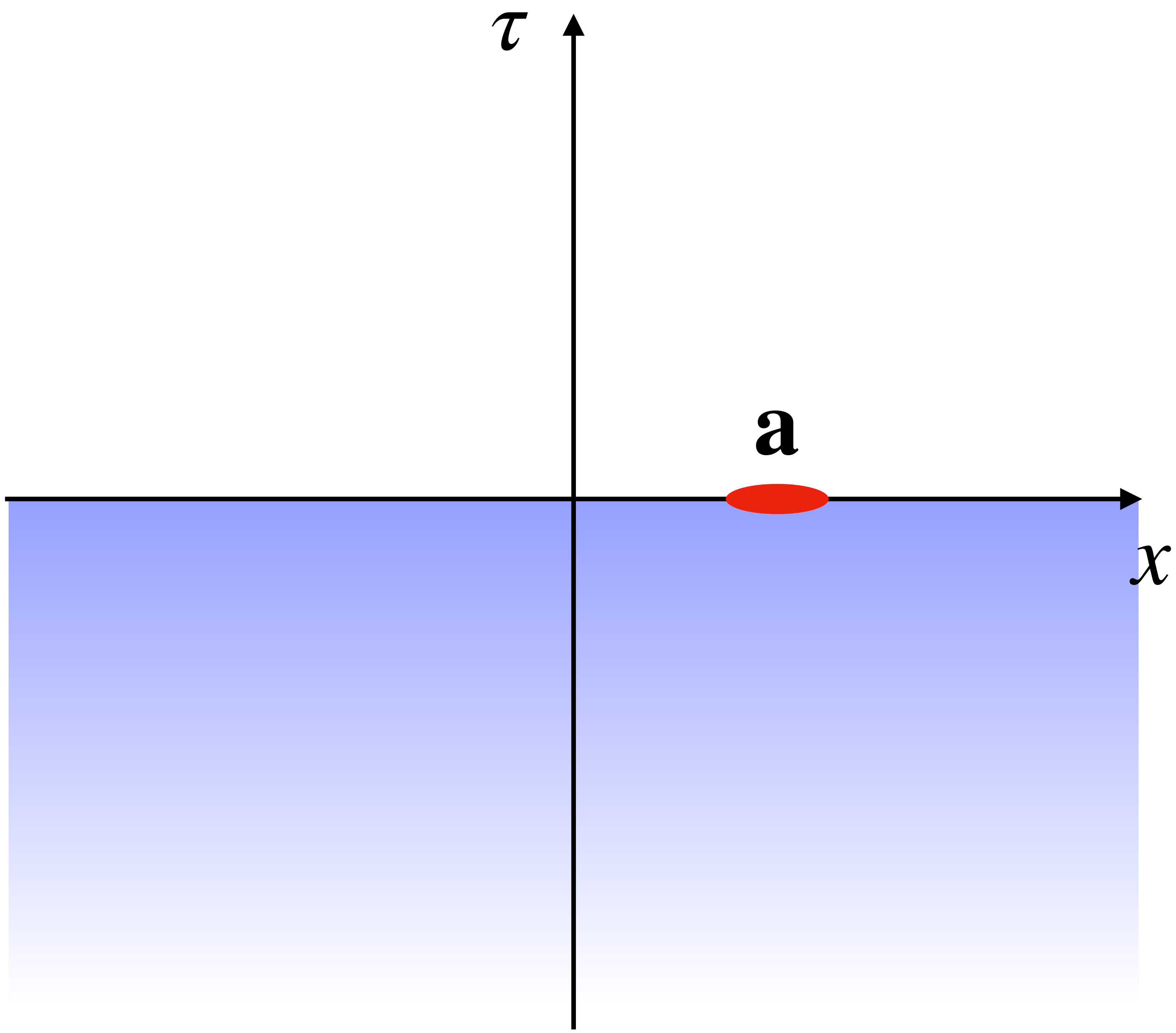
- Next, let us consider a state $\mathbf{a}\,|\,\Omega\rangle$ with $\mathbf{a} \in \mathfrak{A}_r$.
- We assume that the local operator is given by fields without smearing in time.
- Then the state can be given by a path integral on the lower half-space with operator $\mathbf{a}$ inserted on the right half of the boundary.



A FUNDAMENTAL EXAMPLE

II. Path integral approach

- Consider $\Delta_{\Omega}^{\alpha}\mathbf{a}\,|\,\Omega\rangle = \exp(2\pi\alpha K_{\ell})\exp(-2\pi\alpha K_{r})\mathbf{a}\,|\,\Omega\rangle$



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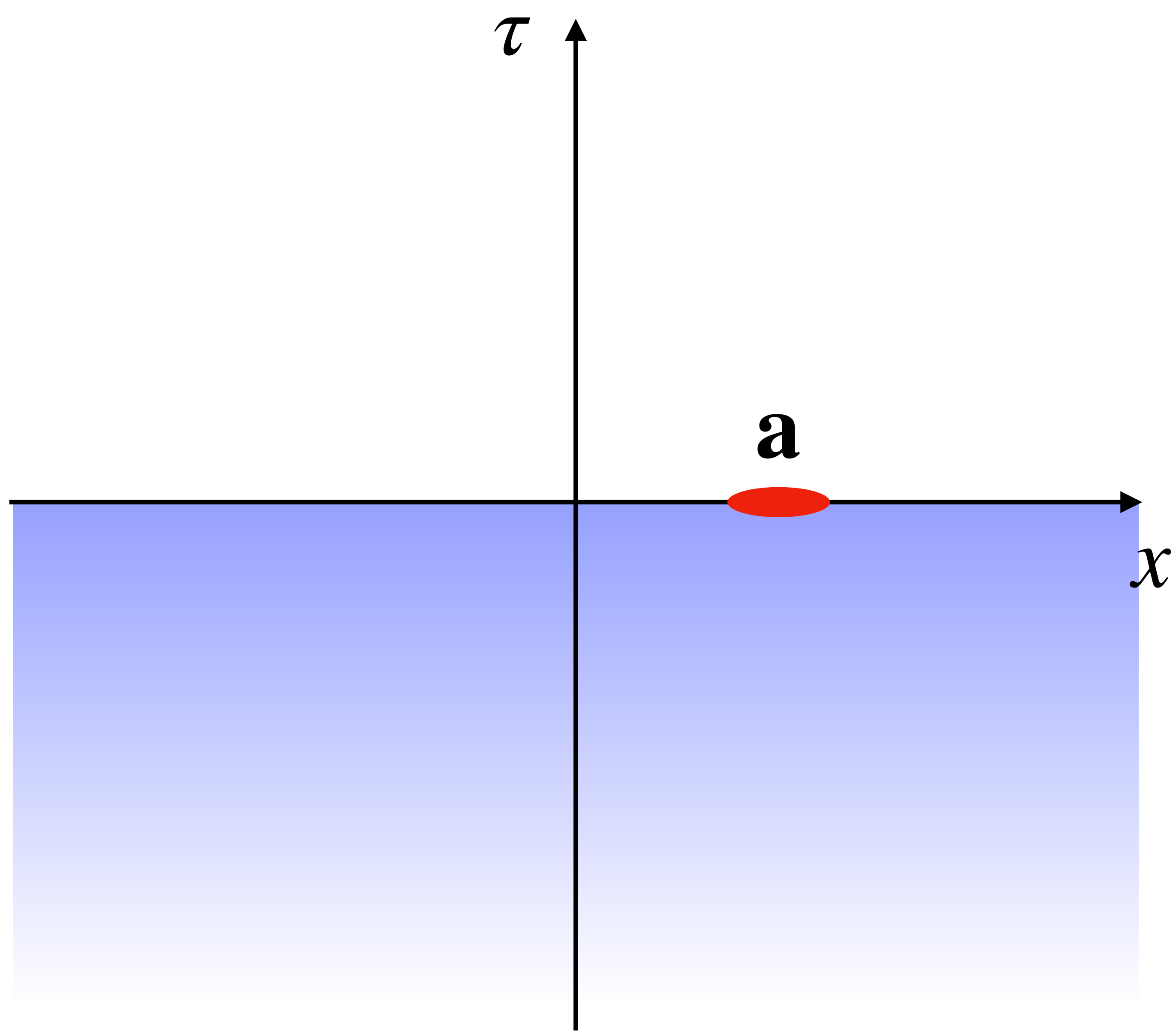

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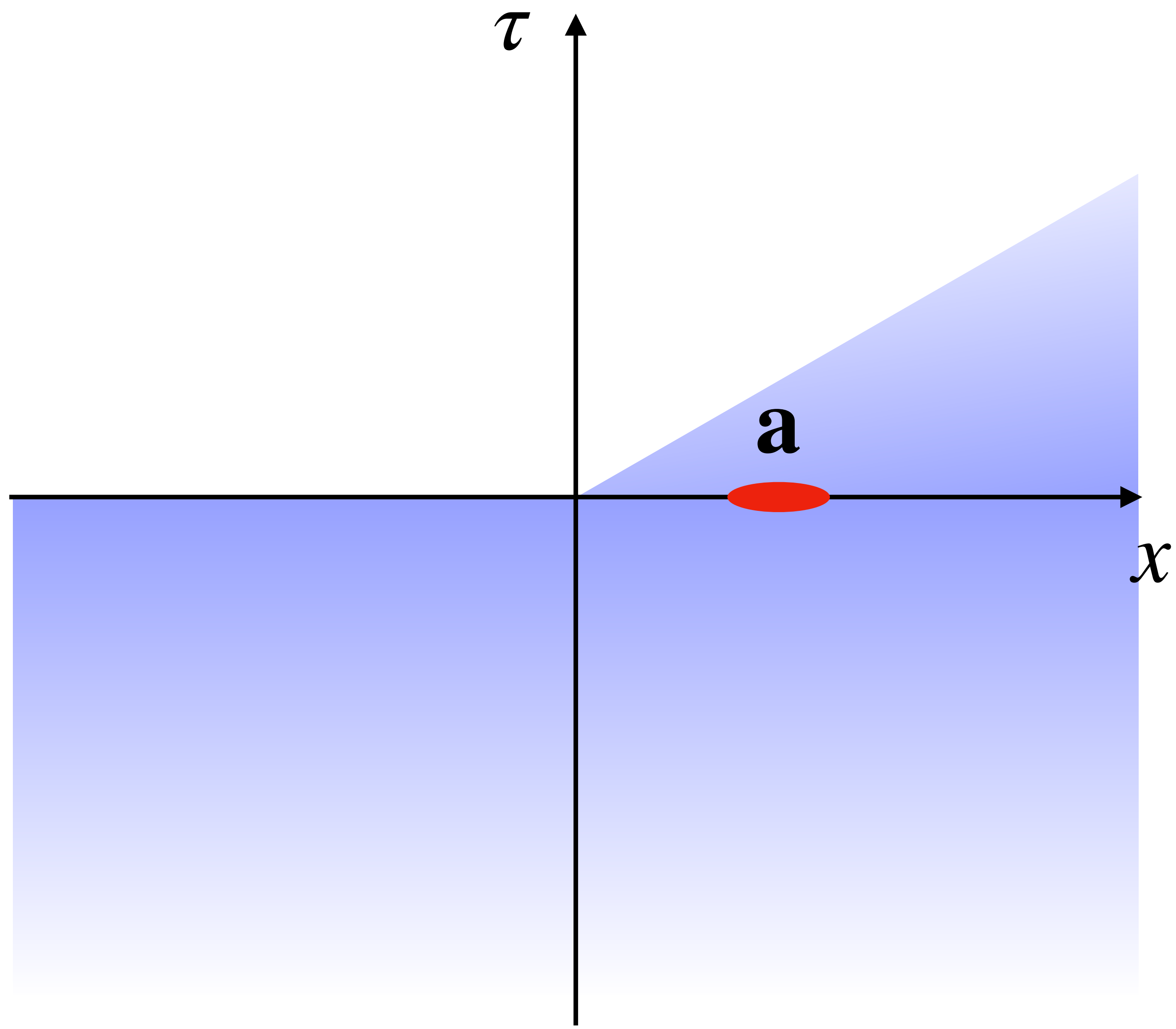

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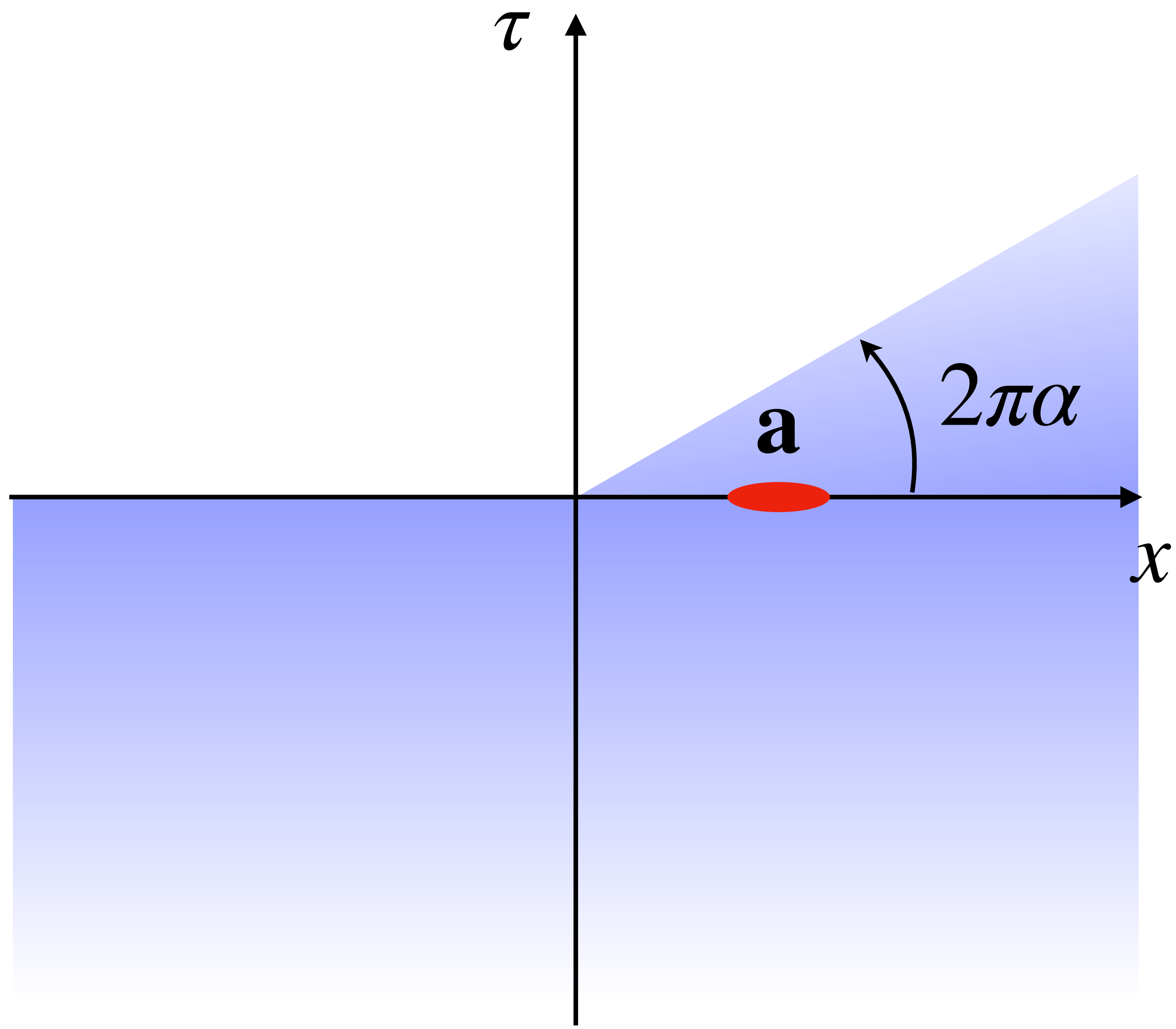

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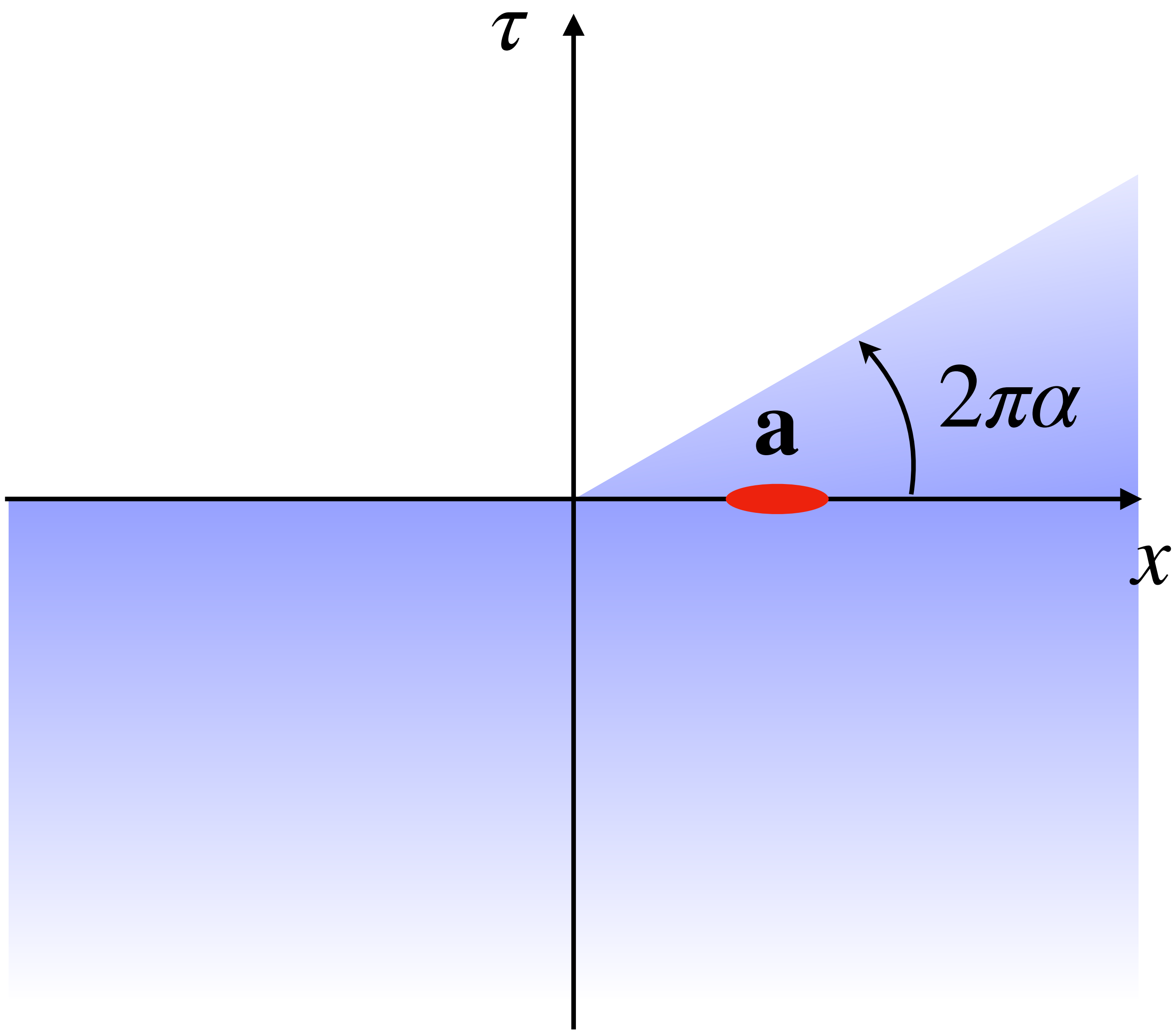

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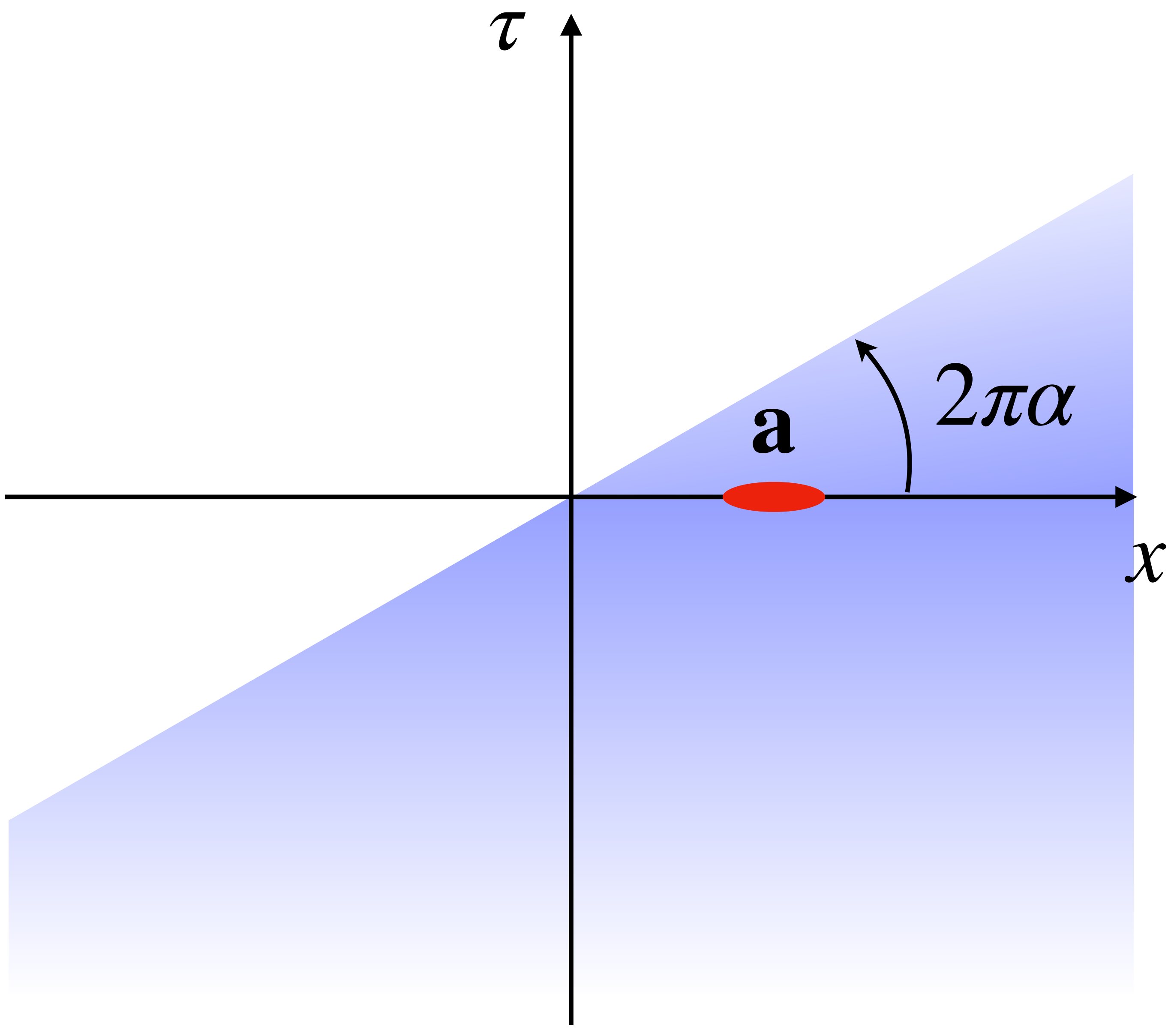

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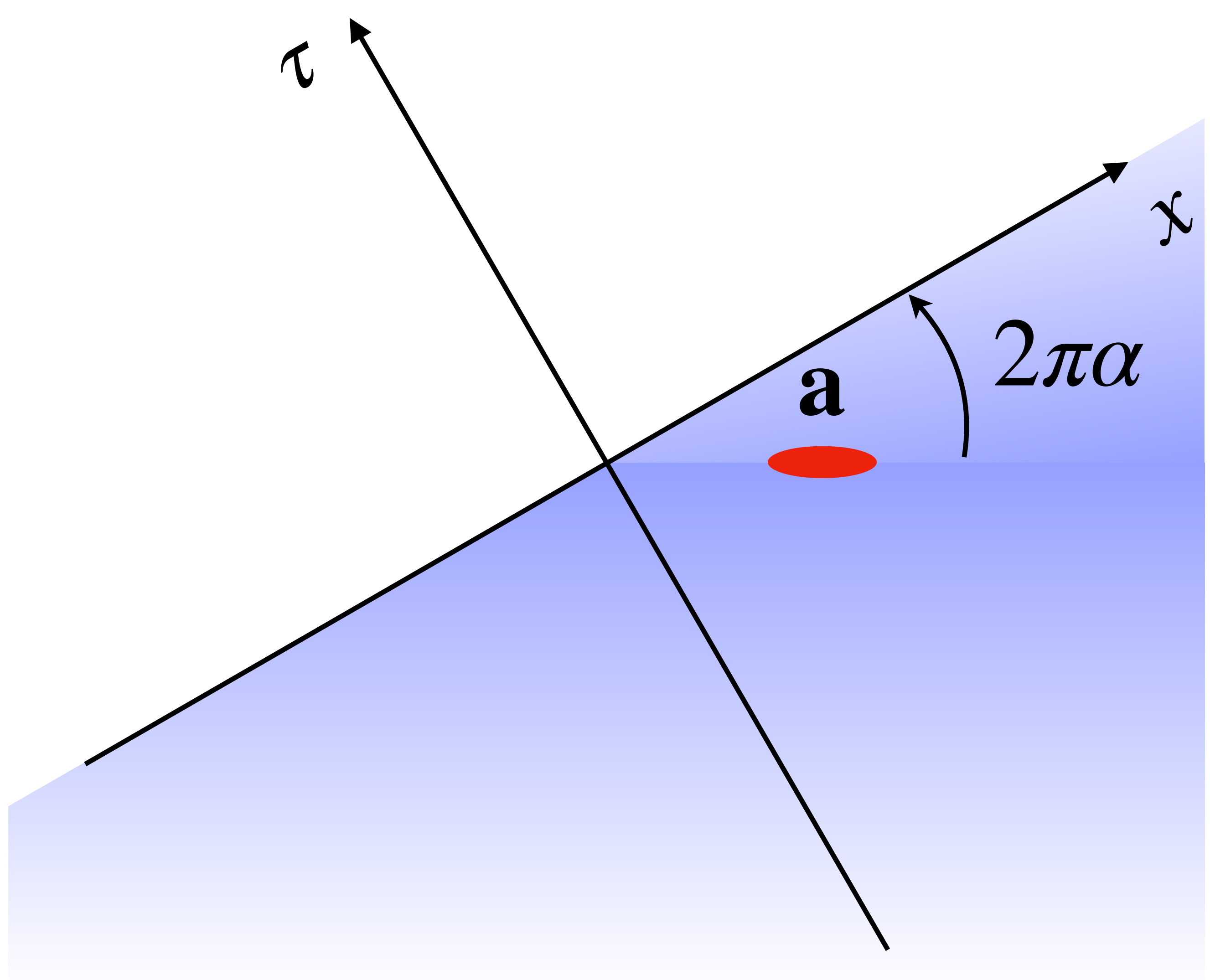

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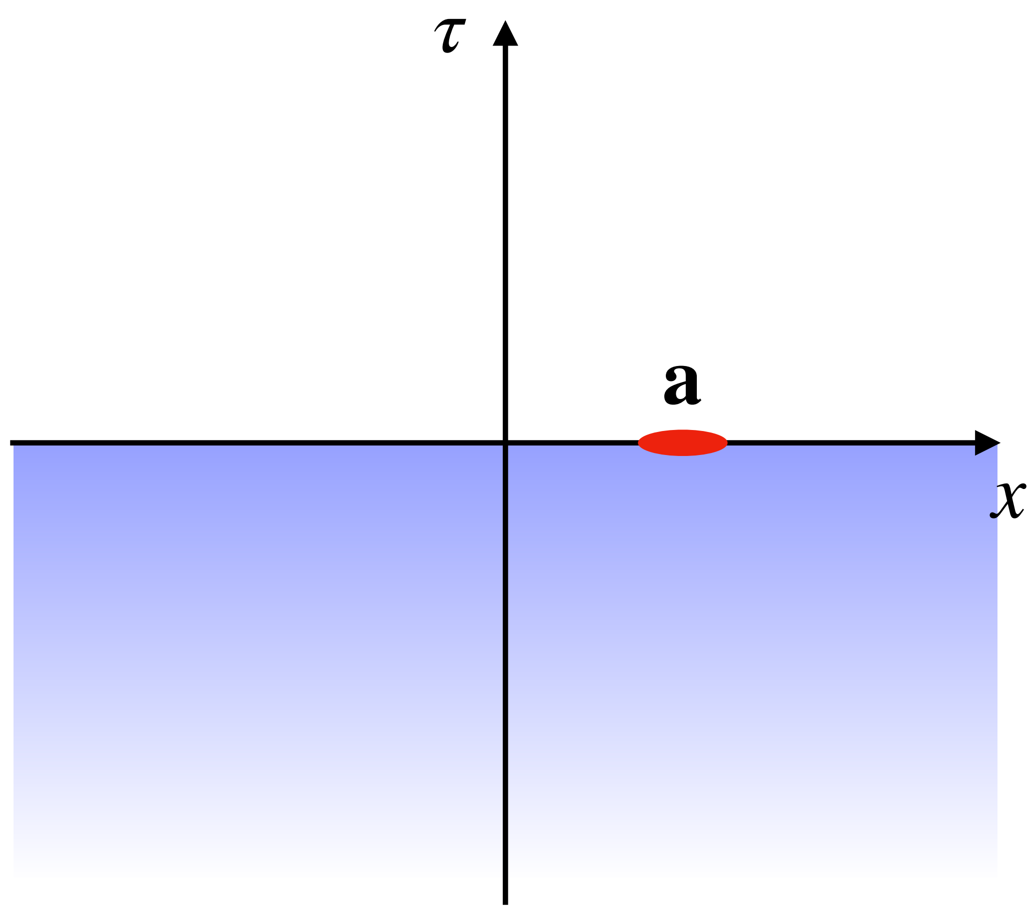

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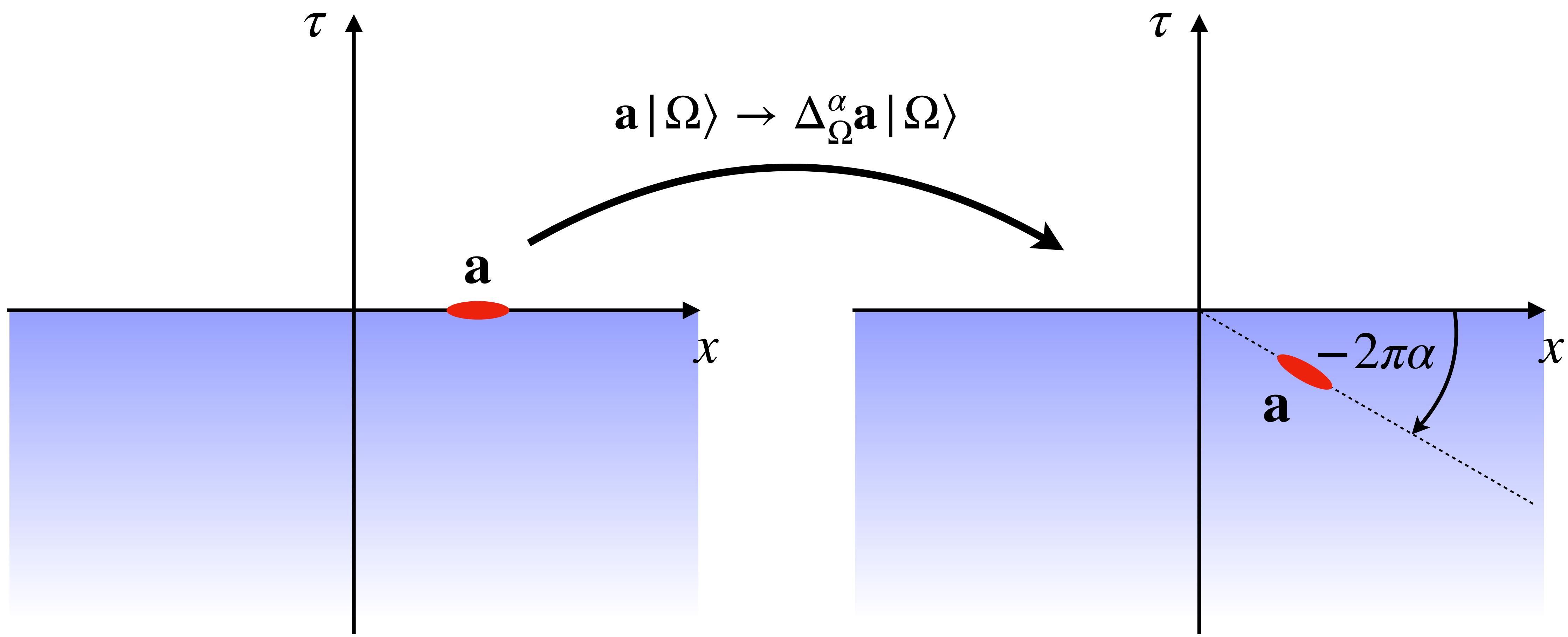
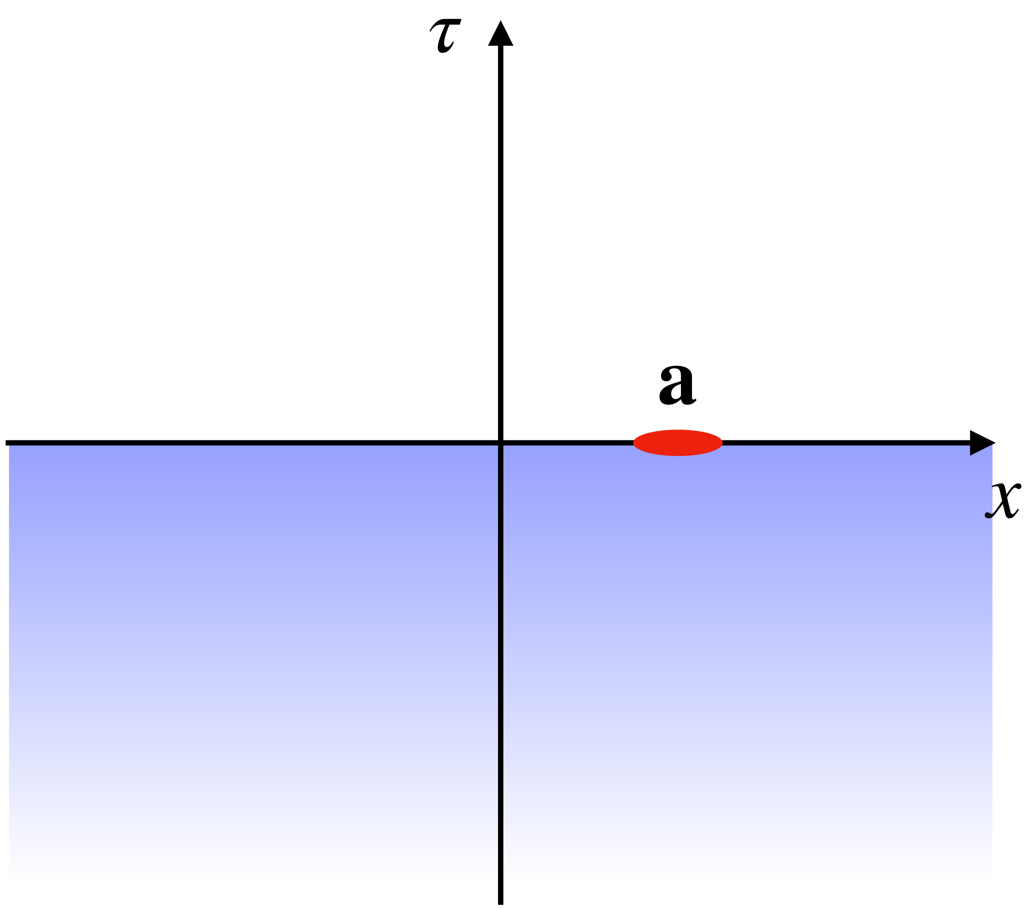


A FUNDAMENTAL EXAMPLE

II. Path integral approach

- If $\alpha = 1/2$, one has $\Delta_{\Omega}^{\alpha}\mathbf{a}\,|\,\Omega\rangle = \exp(\pi K_{\ell})\exp(-\pi K_{r})\mathbf{a}\,|\,\Omega\rangle$



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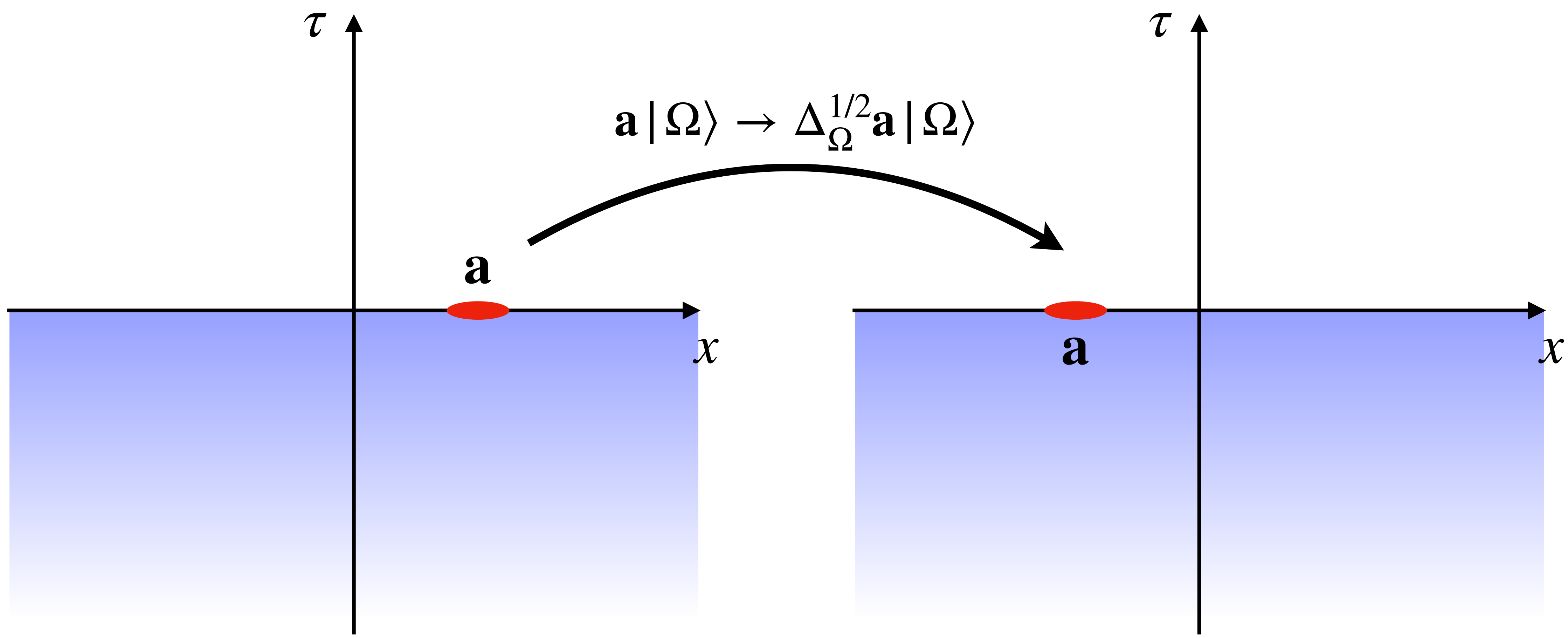

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- If $\alpha = 1/2$, one has $\Delta_{\Omega}^{1/2}\mathbf{a}\,|\,\Omega\rangle = \exp(\pi K_{\ell})\exp(-\pi K_{r})\mathbf{a}\,|\,\Omega\rangle$
- So $\tilde{\mathbf{a}} = \Delta_{\Omega}^{1/2}\mathbf{a}$ is a local operator in $\mathfrak{A}_{\ell}$.
- One can not go to the region $\alpha > 1/2$, otherwise the operator $\mathbf{a}$ will be removed from the path integral.
- So Δ_{Ω}^{iz} is holomorphic in $-1/2 < \mathbf{Im}z < 0$ and continuous on the boundary of this strip.

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- Now we determine the modular conjugation operator J_Ω.

$$S_\Omega = J_\Omega \Delta_\Omega^{1/2}$$

$$\mathbf{a}^\dagger \,|\,\Omega\rangle = S_\Omega \mathbf{a}\,|\,\Omega\rangle = J_\Omega \Delta_\Omega^{1/2} \mathbf{a}\,|\,\Omega\rangle = J_\Omega \tilde{\mathbf{a}}\,|\,\Omega\rangle$$

- For simplicity, we consider a QFT of single Hermitian scalar field $\varphi(t, x, \mathbf{y})$.

- It suffices to check $S_\Omega \varphi(0,x,\mathbf{y})\,|\,\Omega\rangle$ and $S_\Omega \dot{\varphi}(0,x,\mathbf{y})\,|\,\Omega\rangle$.

A FUNDAMENTAL EXAMPLE

II. Path integral approach

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A FUNDAMENTAL EXAMPLE

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- For simplicity, we consider a QFT of single Hermitian scalar field $\varphi(t, x, \mathbf{y})$.

- Because they are Hermitian field, whose conjugation are themselves

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- There is a typo in Eq. (5.13) in the original paper.

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- This result means $J_\Omega : \; t \to -t, \; x \to -x, \; \mathbf{y} \to \mathbf{y}$

- So the modular conjugation operator is just the CR_xT transformation.

$$J_\Omega = CRT$$

A FUNDAMENTAL EXAMPLE

II. Path integral approach

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- The CRT is a universal symmetry of relativistic quantum field theory, while there is no symmetry corresponding to RT.

A FUNDAMENTAL EXAMPLE

II. Path integral approach

- We verify the deeper properties of the modular operator Δ_Ω and the modular conjugation J_Ω explicitly:
    - Δ_Ω^{is}: Lorentz boost with real boost factor $2\pi s$;
    - $\Delta_\Omega^{is}: \ \mathfrak{A}_\ell \to \mathfrak{A}_\ell$ and $\Delta_\Omega^{is}: \ \mathfrak{A}_r \to \mathfrak{A}_r$ are automorphisms;
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  - $J_\Omega = CRT$ and $J_\Omega: \ \mathfrak{A}_\ell \leftrightarrow \mathfrak{A}_r$ exchanges the two wedge algebras.
- In Takesaki-Tomita theory, the modular conjugation J_Ω exchanges the algebra with its commutant , so

$$\mathfrak{A}_\ell' = \mathfrak{A}_r, \quad \mathfrak{A}_r' = \mathfrak{A}_\ell$$

- Thus the Haag duality for complementary Rindler spacetime is proved.

A FUNDAMENTAL EXAMPLE

III. The approach of Bisognano and Wichmann

- The path integral method is extremely illustrating and gives the right result, but it is not rigorous.

- The Hilbert space of quantum field theory can not be factorized as $\mathscr{H}_\ell \otimes \mathscr{H}_r$!

- In the rigorous proof (Bisognano and Wichmann, 1975, 1976), one uses holomorphy.

A FUNDAMENTAL EXAMPLE

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Arthur Strong
Wightman
(1922/05/30-2013/01/13)

Raymond Frederick
"Ray" Streater
(1936/04/21-)

A FUNDAMENTAL EXAMPLE

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- Denote the vacuum expectation values (Wightman functions) by $\mathscr{W}(x_1, x_2, \cdots, x_n) = \langle \Omega \,|\, \varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_n(x_n) \,|\, \Omega \rangle$. By translation symmetry, one has $\mathscr{W}(x_1, x_2, \cdots, x_n) = W(\xi_1, \xi_2, \cdots, \xi_{n-1})$, where $\xi_j = x_j - x_{j+1}$. Then there exist (the domain of holomorphy being $\eta_j \in \mathbf{V}_+$) holomorphic function $\mathbf{W}(\xi_1 - i\eta_1, \cdots, \xi_{n-1} - i\eta_{n-1})$, such that

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$$W(\xi_1, \cdots, \xi_{n-1}) = \lim_{\eta_1, \cdots, \eta_{n-1} \to 0^+} \mathbf{W}(\xi_1 - i\eta_1, \cdots, \xi_{n-1} - i\eta_{n-1})$$

A FUNDAMENTAL EXAMPLE

III. The approach of Bisognano and Wichmann

- Since $J_\Omega = CRT$ certainly acts as $J_\Omega \varphi(0, x, \mathbf{y}) J_\Omega^{-1} = \varphi(0, -x, \mathbf{y})$ and $J_\Omega \dot{\varphi}(0, x, \mathbf{y}) J_\Omega^{-1} = -\dot{\varphi}(0, -x, \mathbf{y})$, to determine Δ_Ω and S_Ω, one has to justify the claim that for $\mathbf{a} \in \mathfrak{A}_r$

$$\exp(-2\pi K)\mathbf{a}\,|\,\Omega\rangle = \tilde{\mathbf{a}}\,|\,\Omega\rangle$$

- Here $\tilde{\mathbf{a}}$ is obtained from $\mathbf{a}$ by $(t, x, \mathbf{y}) \to (-t, -x, \mathbf{y})$.

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- Here $\tilde{\mathbf{a}}$ is obtained from $\mathbf{a}$ by $(t, x, \mathbf{y}) \to (-t, -x, \mathbf{y})$.

- We check it for $\mathbf{a} = \varphi(t_1, x_1, \mathbf{y}_1)\varphi(t_2, x_2, \mathbf{y}_2)\cdots\varphi(t_n, x_n, \mathbf{y}_n)$., where the points $p_1 = (t_1, x_1, \mathbf{y}_1), p_2 = (t_2, x_2, \mathbf{y}_2), \cdots, p_n = (t_n, x_n, \mathbf{y}_n)$ are all in the right wedge $\mathscr{U}_r$.

- So we have $x_j > |\,t_j\,|$.

A FUNDAMENTAL EXAMPLE

III. The approach of Bisognano and Wichmann

- We can take p_j to be spacelike separated from each other.

- Then the field operators commute, we can order them so that $x_j \geqslant x_i$ for $j > i$.

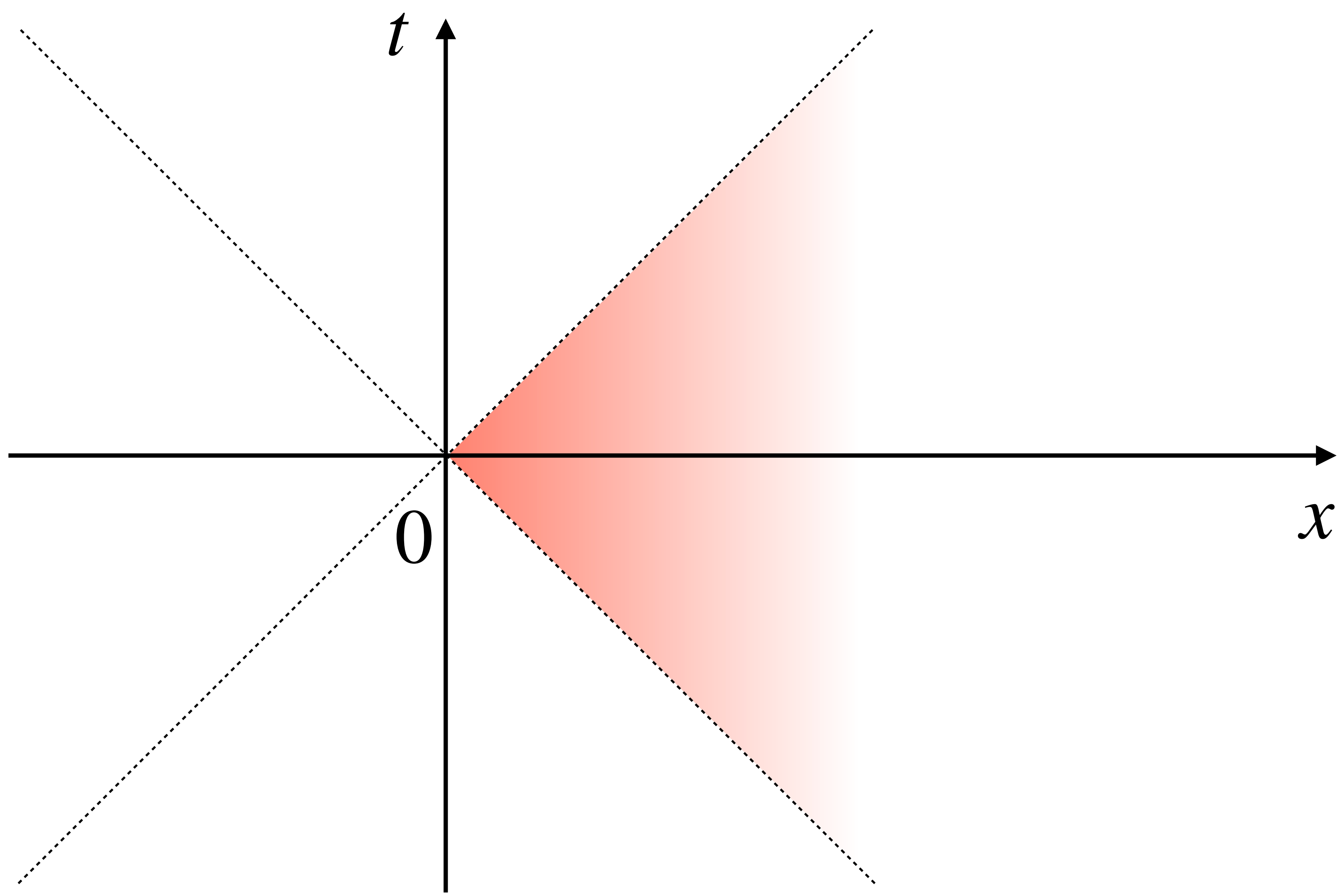
- Even more specially, we can restrict to $x_j - x_i > |t_j - t_i|$ for $j > i$.

- The states $\mathbf{a}|\Omega\rangle$ with $\mathbf{a}$ of this type are dense in $\mathscr{H}$. (The proof is similar to that for Reeh-Schlieder theorem.)

A FUNDAMENTAL EXAMPLE

III. The approach of Bisognano and Wichmann

- We restrict to $x_j - x_i > |t_j - t_i|$ for $j > i$.



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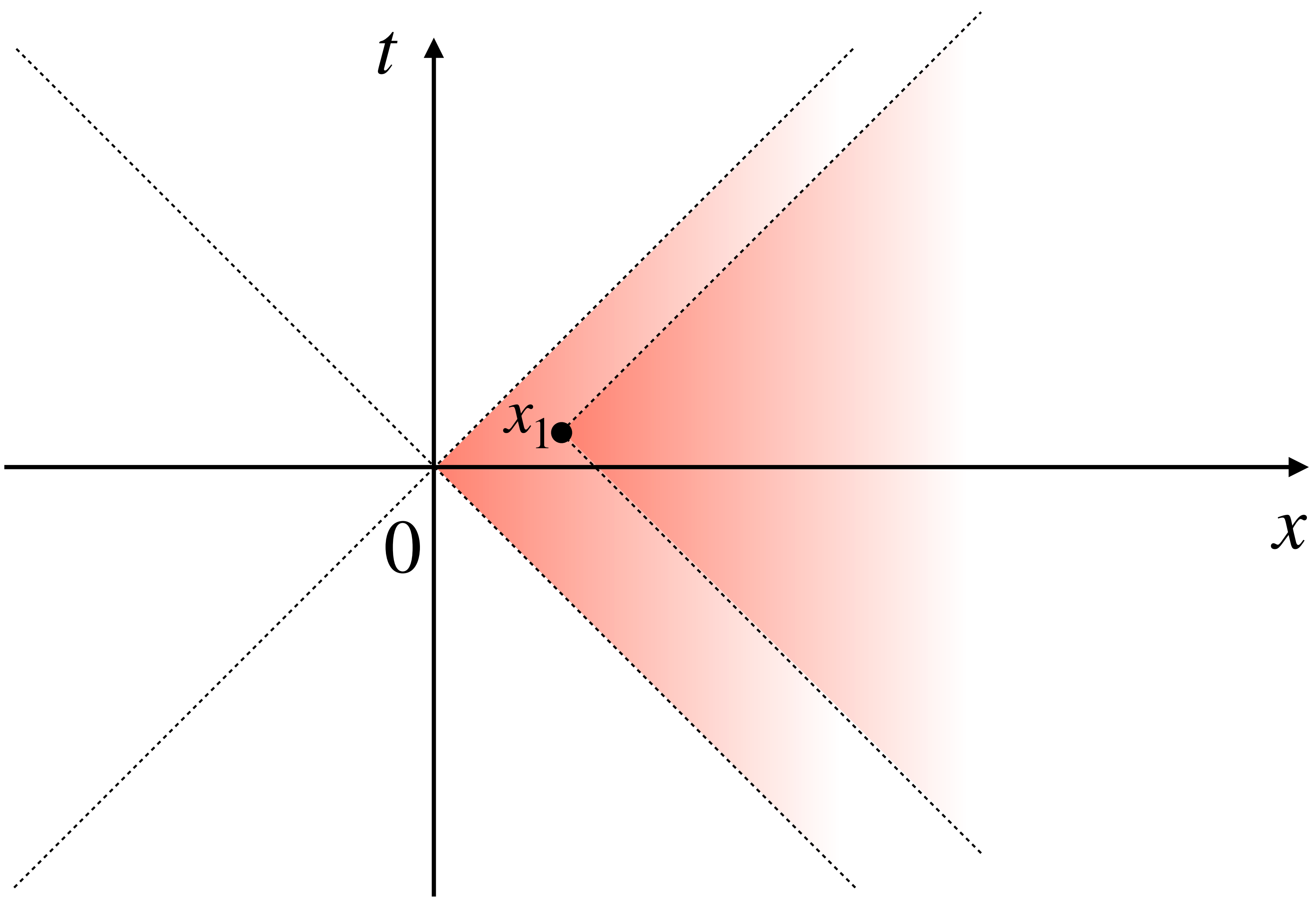

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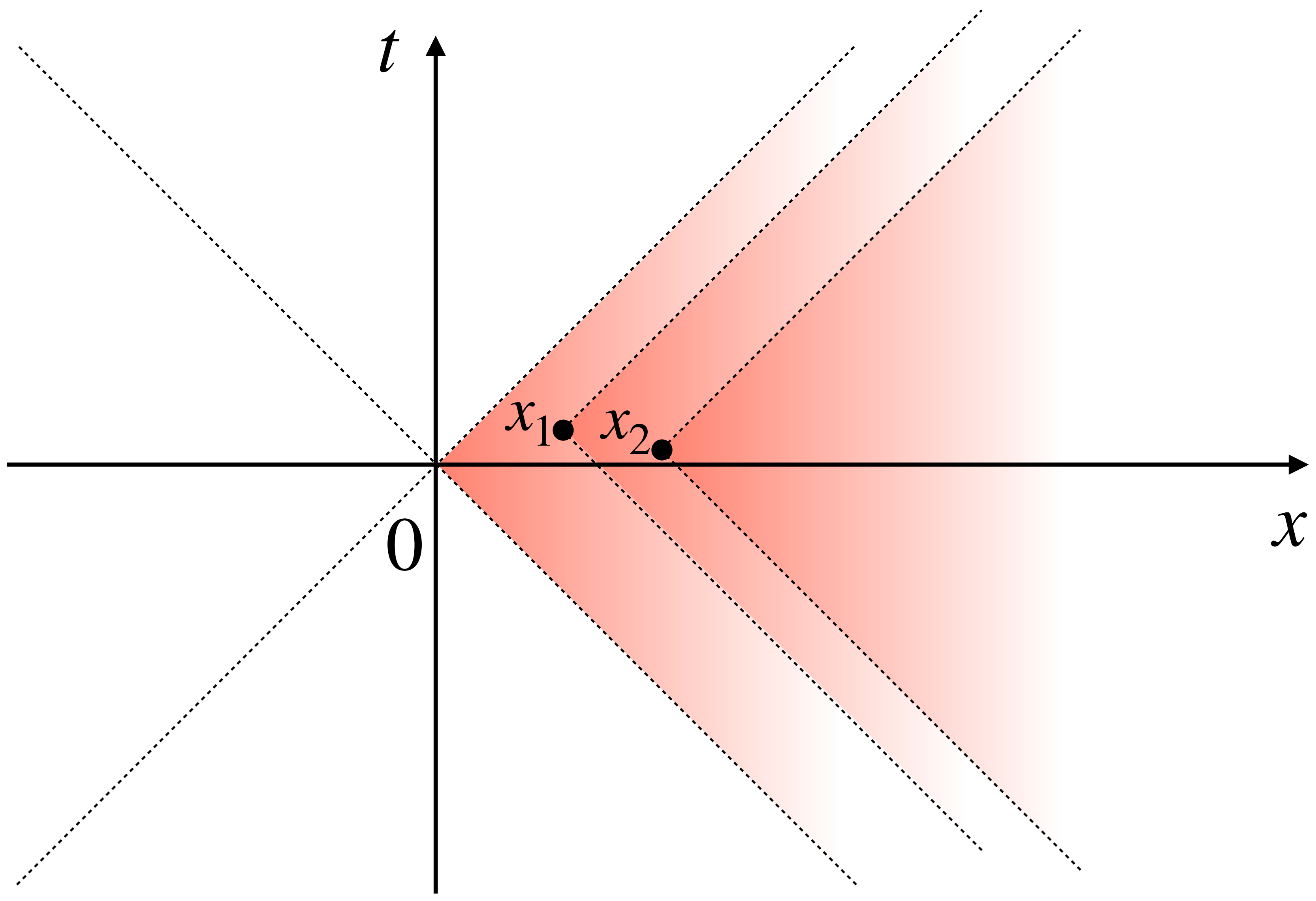

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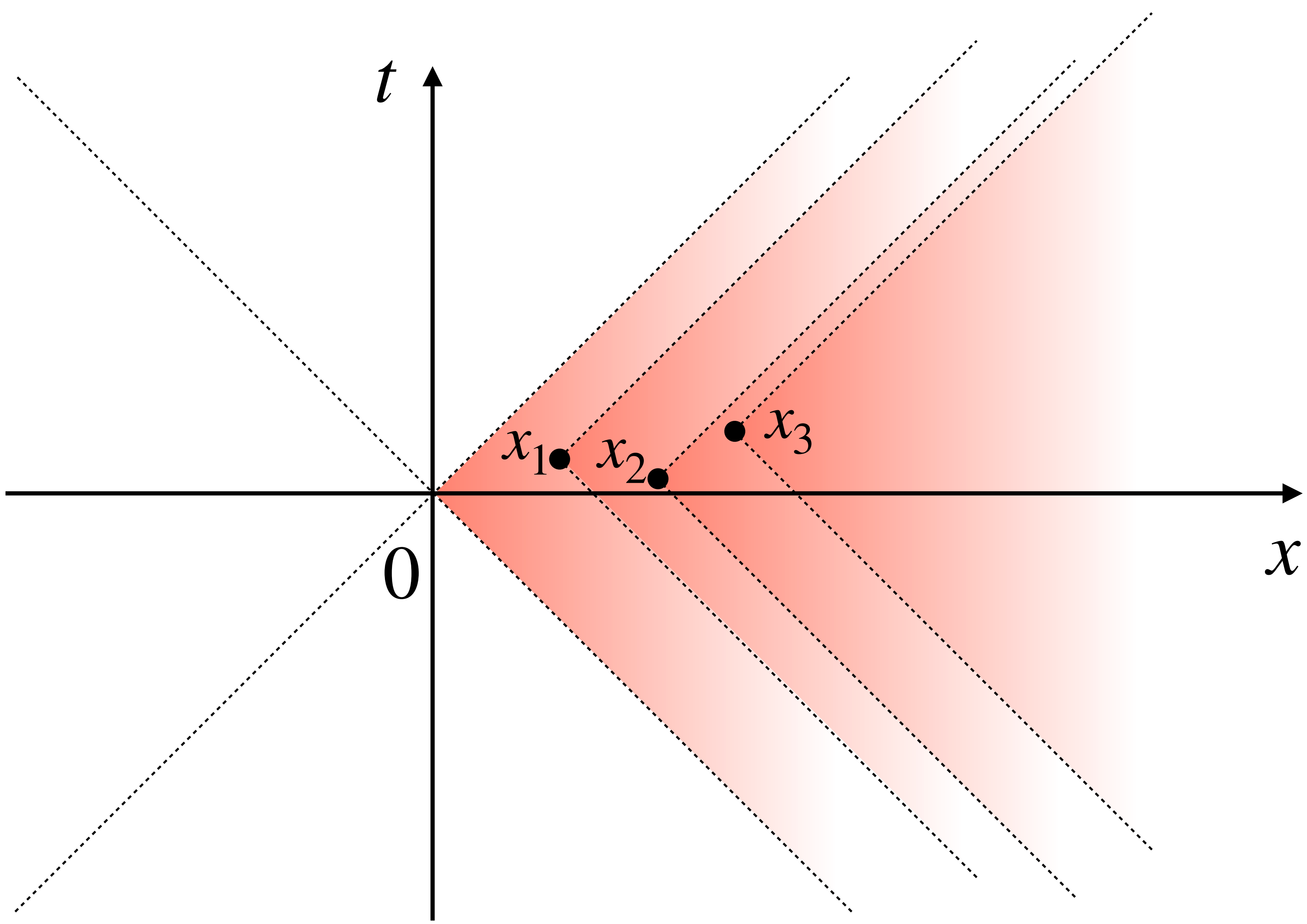

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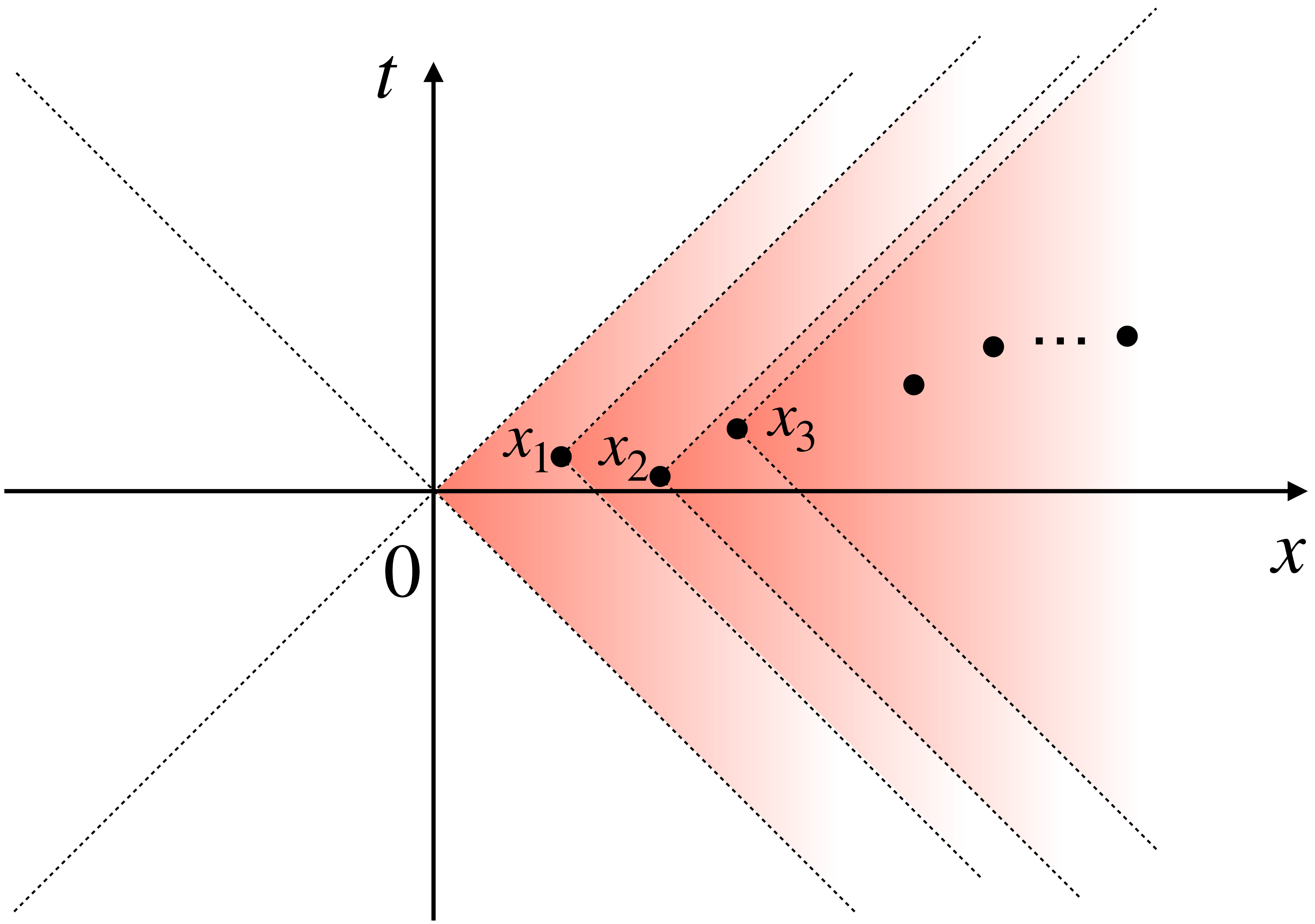

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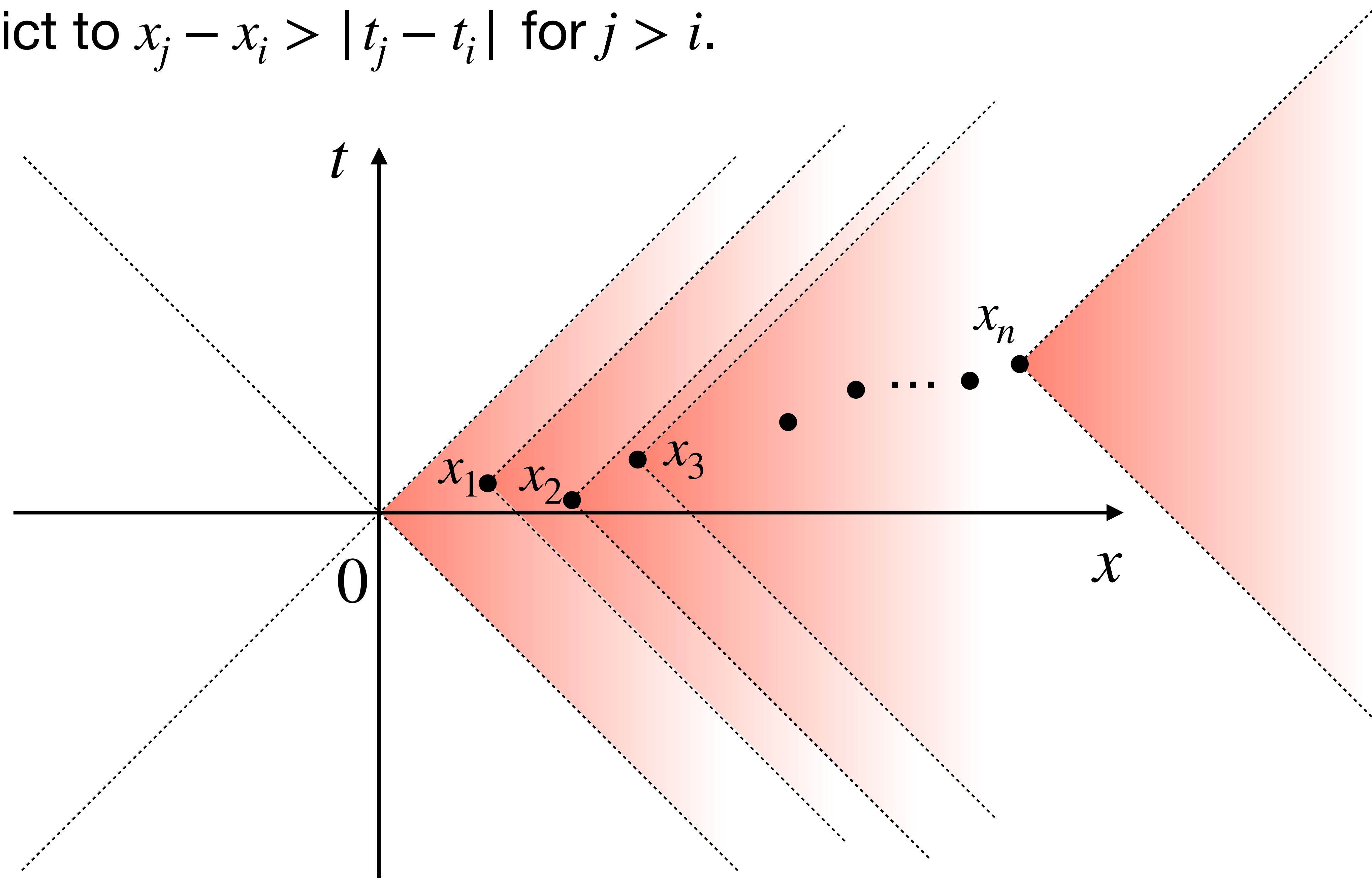

A FUNDAMENTAL EXAMPLE

III. The approach of Bisognano and Wichmann

- We first check the Lorentz boost $\exp(-2\pi i s K)$ with real boost factor s.

- It is a unitary transformation on any state $\mathbf{a}\,|\,\Omega\rangle$.

- Because it is a Poincare transformation, its action is given by

$$\exp(2\pi i\eta K)\varphi(\mathbf{x})\exp(-2\pi i\eta K) = \varphi(\mathbf{x}(\eta))$$

- The $\mathbf{x}(\eta)$ is the Lorentz transformation of the spacetime coordinate

$$\mathbf{x}(\eta) = \begin{pmatrix} t(\eta) \\ x(\eta) \end{pmatrix} = \begin{pmatrix} \cosh(2\pi\eta) & \sinh(2\pi\eta) \\ \sinh(2\pi\eta) & \cosh(2\pi\eta) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

A FUNDAMENTAL EXAMPLE

III. The approach of Bisognano and Wichmann

- Because the vacuum is invariant under Poincare transformation, we have $K|\Omega\rangle = 0$.

$$\exp(2\pi i\eta K)\mathbf{a}|\Omega\rangle = \exp(2\pi i\eta K)\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)\cdots\varphi(\mathbf{x}_n)|\Omega\rangle$$

$$= \exp(2\pi i\eta K)\varphi(\mathbf{x}_1)\exp(-2\pi i\eta K)\ \exp(2\pi i\eta K)\varphi(\mathbf{x}_2)\exp(-2\pi i\eta K)\cdots$$
$$\cdots\exp(2\pi i\eta K)\varphi(\mathbf{x}_n)\exp(-2\pi i\eta K)\exp(2\pi i\eta K)|\Omega\rangle$$

$$= \varphi(\mathbf{x}_1(\eta))\varphi(\mathbf{x}_2(\eta))\cdots\varphi(\mathbf{x}_n(\eta))|\Omega\rangle$$

- We want to analytically continue this formula in η to $\eta = i/2$ because

$$\mathbf{x}(i/2) = \begin{pmatrix} \cosh(i\pi) & \sinh(i\pi) \\ \sinh(i\pi) & \cosh(i\pi) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = -\mathbf{x}$$

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- So we need to show that when $0 < \mathbf{Im}\eta < 1/2$ the imaginary part of $\mathbf{x}_{j+1} - \mathbf{x}_j$ is future timelike.

A FUNDAMENTAL EXAMPLE

III. The approach of Bisognano and Wichmann

- For $\eta = a + ib$,

$$\cosh(2\pi(a+ib)) = \cos(2\pi b)\cosh(2\pi a) + i\sin(2\pi b)\sinh(2\pi a)$$
$$\sinh(2\pi(a+ib)) = \cos(2\pi b)\sinh(2\pi a) + i\sin(2\pi b)\cosh(2\pi a)$$

$$\begin{aligned}
\mathbf{x}(a+ib) &= \begin{pmatrix} \cos(2\pi b)\cosh(2\pi a) + i\sin(2\pi b)\sinh(2\pi a) & \cos(2\pi b)\sinh(2\pi a) + i\sin(2\pi b)\cosh(2\pi a) \\ \cos(2\pi b)\sinh(2\pi a) + i\sin(2\pi b)\cosh(2\pi a) & \cos(2\pi b)\cosh(2\pi a) + i\sin(2\pi b)\sinh(2\pi a) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\
&= \begin{pmatrix} \cos(2\pi b)[t\cosh(2\pi a) + x\sinh(2\pi a)] + i\sin(2\pi b)[t\sinh(2\pi a) + x\cosh(2\pi a)] \\ \cos(2\pi b)[t\sinh(2\pi a) + x\cosh(2\pi a)] + i\sin(2\pi b)[t\cosh(2\pi a) + x\sinh(2\pi a)] \end{pmatrix} \\
&= \cos(2\pi b)\begin{pmatrix} t\cosh(2\pi a) + x\sinh(2\pi a) \\ t\sinh(2\pi a) + x\cosh(2\pi a) \end{pmatrix} + i\sin(2\pi b)\begin{pmatrix} t\sinh(2\pi a) + x\cosh(2\pi a) \\ t\cosh(2\pi a) + x\sinh(2\pi a) \end{pmatrix}
\end{aligned}$$

$$\therefore\ \mathbf{Im}(\mathbf{x}_{j+1}(a+ib) - \mathbf{x}_j(a+ib)) = \sin(2\pi b)\begin{pmatrix} \sinh(2\pi a) & \cosh(2\pi a) \\ \cosh(2\pi a) & \sinh(2\pi a) \end{pmatrix} \begin{pmatrix} t_{j+1} - t_j \\ x_{j+1} - x_j \end{pmatrix}$$

$$\begin{aligned}
\therefore\ |\mathbf{Im}(\mathbf{x}_{j+1}(a+ib) - \mathbf{x}_j(a+ib))| &= \sin^2(2\pi b)[\cosh^2(2\pi a) - \sinh^2(2\pi a)][(x_{j+1} - x_j)^2 - (t_{j+1} - t_j)^2] \\
&= \sin^2(2\pi b)[(x_{j+1} - x_j)^2 - (t_{j+1} - t_j)^2] > 0
\end{aligned}$$

A FUNDAMENTAL EXAMPLE

III. The approach of Bisognano and Wichmann

- Because the imaginary part of the coordinates are 0, we have proved that for $\eta = a + ib$ and $0 < \mathbf{Im}\eta < 1/2$, $\mathbf{Im}(\mathbf{x}_{j+1} - \mathbf{x}_j)$ is timelike.

- Because $0 < b < 1/2$ and $x_{j+1} - x_j > |t_{j+1} - t_j|$, the time component of $\mathbf{Im}(\mathbf{x}_{j+1} - \mathbf{x}_j)$ is

$$\sin(2\pi b)[(x_{j+1} - x_j)\cosh(2\pi a) + (t_{j+1} - t_j)\sinh(2\pi a)] > 0$$

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$$\sin(2\pi b)[(x_{j+1} - x_j)\cosh(2\pi a) + (t_{j+1} - t_j)\sinh(2\pi a)] > 0$$

- So $\varphi(\mathbf{x}_1(\eta))\varphi(\mathbf{x}_2(\eta))\cdots\varphi(\mathbf{x}_n(\eta))|\Omega\rangle$ is holomorphic for $0 < \mathbf{Im}\eta < 1/2$ and continuous up to the boundary at $\mathbf{Im}\eta = 1/2$.

- Then we have $\exp(-2\pi K)\mathbf{a}|\Omega\rangle = \tilde{\mathbf{a}}|\Omega\rangle$, which finishes the proof.

A FUNDAMENTAL EXAMPLE

IV. An accelerating observer

- Unruh’s question: what is seen by an observer undergoing constant acceleration in Minkowski spacetime?

William George "Bill" Unruh
(1945/08/28-)

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$$\therefore \; \left(\frac{dU^0}{d\tau}, \frac{dU^1}{d\tau}, 0, \cdots, 0\right) = \frac{1}{R}\left(U^1, U^0, 0, \cdots, 0\right)$$

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$$\Rightarrow \;\; \begin{cases} U^0(\tau) = \cosh(\tau/R) \\ U^1(\tau) = \sinh(\tau/R) \end{cases} \;\; \Rightarrow \begin{cases} T(\tau) = R\sinh(\tau/R) \\ X(\tau) = R\cosh(\tau/R) \end{cases}$$

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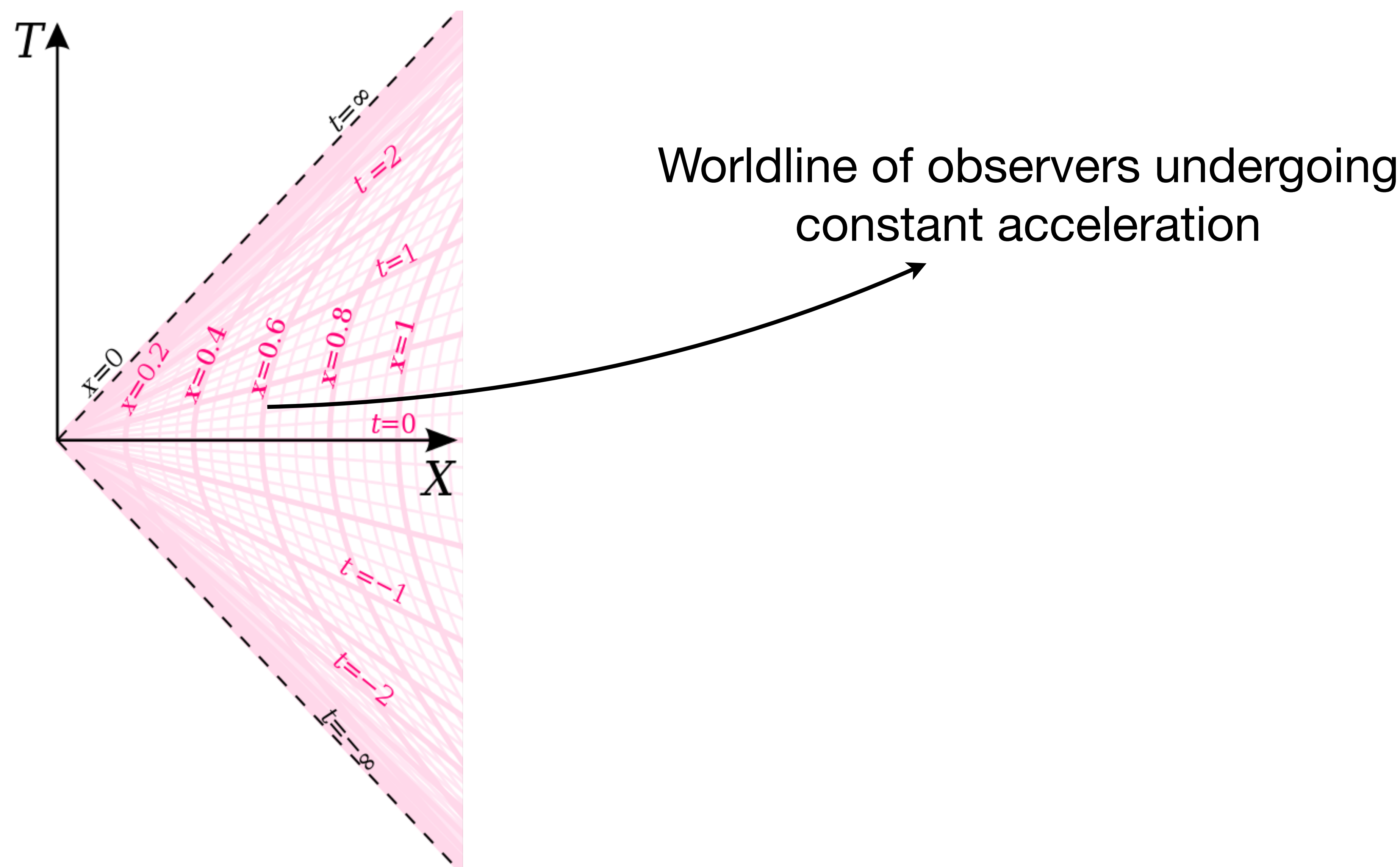

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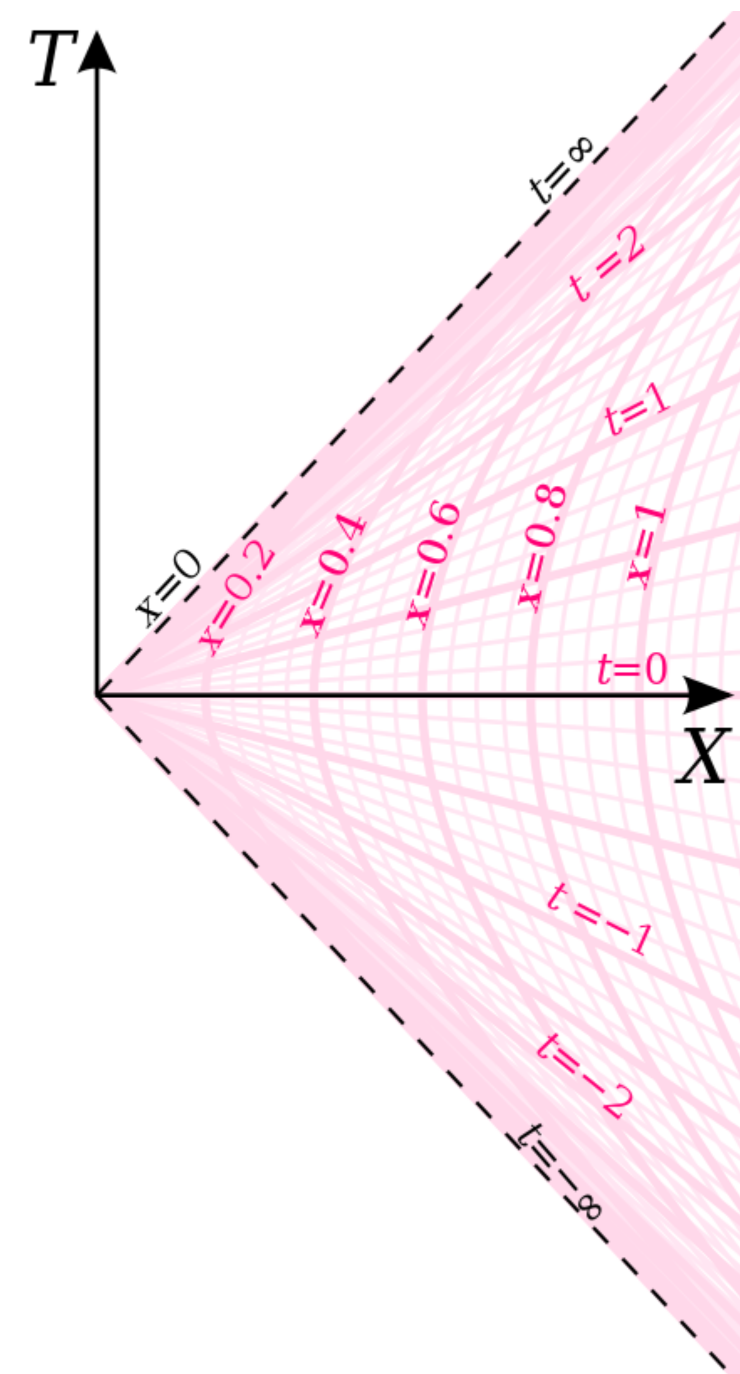
- The observer can probe the vacuum $|\Omega\rangle$ by measuring a local operator $\mathcal{O}$ and its adjoint $\mathcal{O}^\dagger$ along its worldline.

- For simplicity, we consider the two-point functions $\mathcal{O} \cdot \mathcal{O}^\dagger$ with different orders $\langle\Omega|\mathcal{O}(\mathbf{x}(\tau_1))\mathcal{O}^\dagger(\mathbf{x}(\tau_2))|\Omega\rangle$ and $\langle\Omega|\mathcal{O}^\dagger(\mathbf{x}(\tau_2))\mathcal{O}(\mathbf{x}(\tau_1))|\Omega\rangle$.



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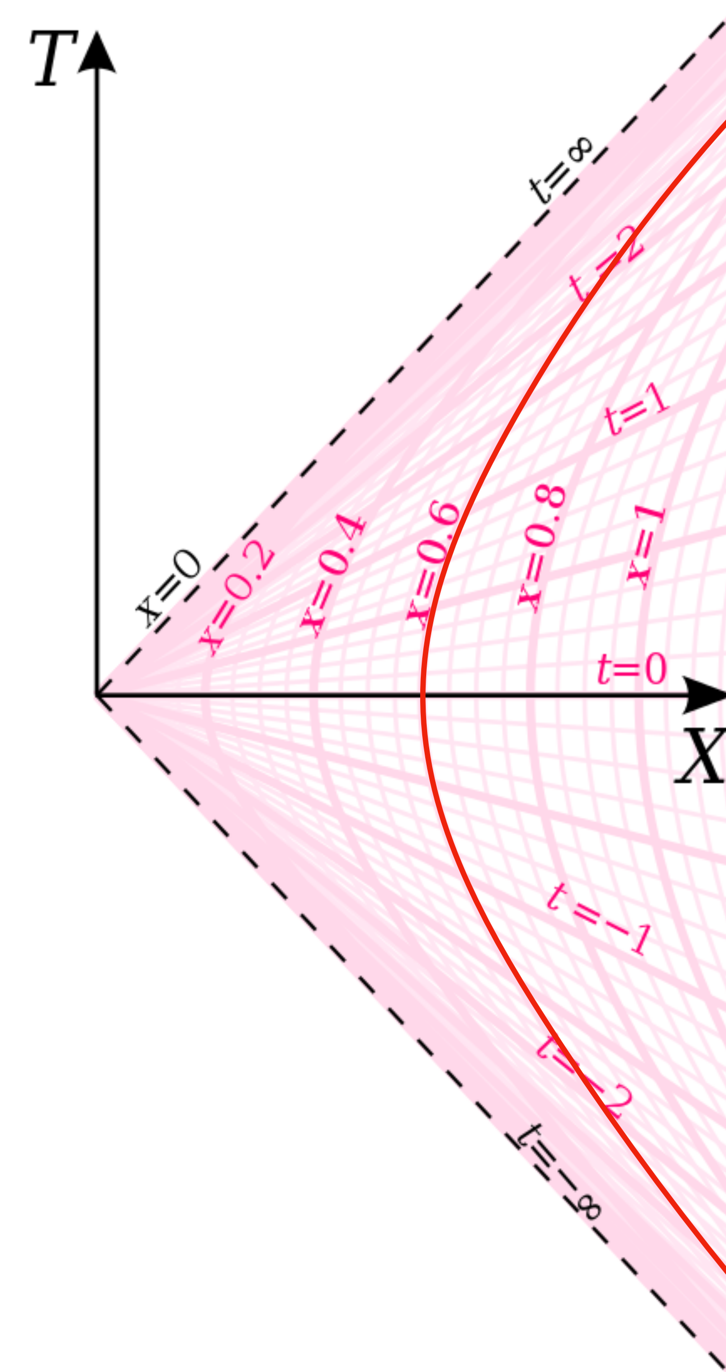

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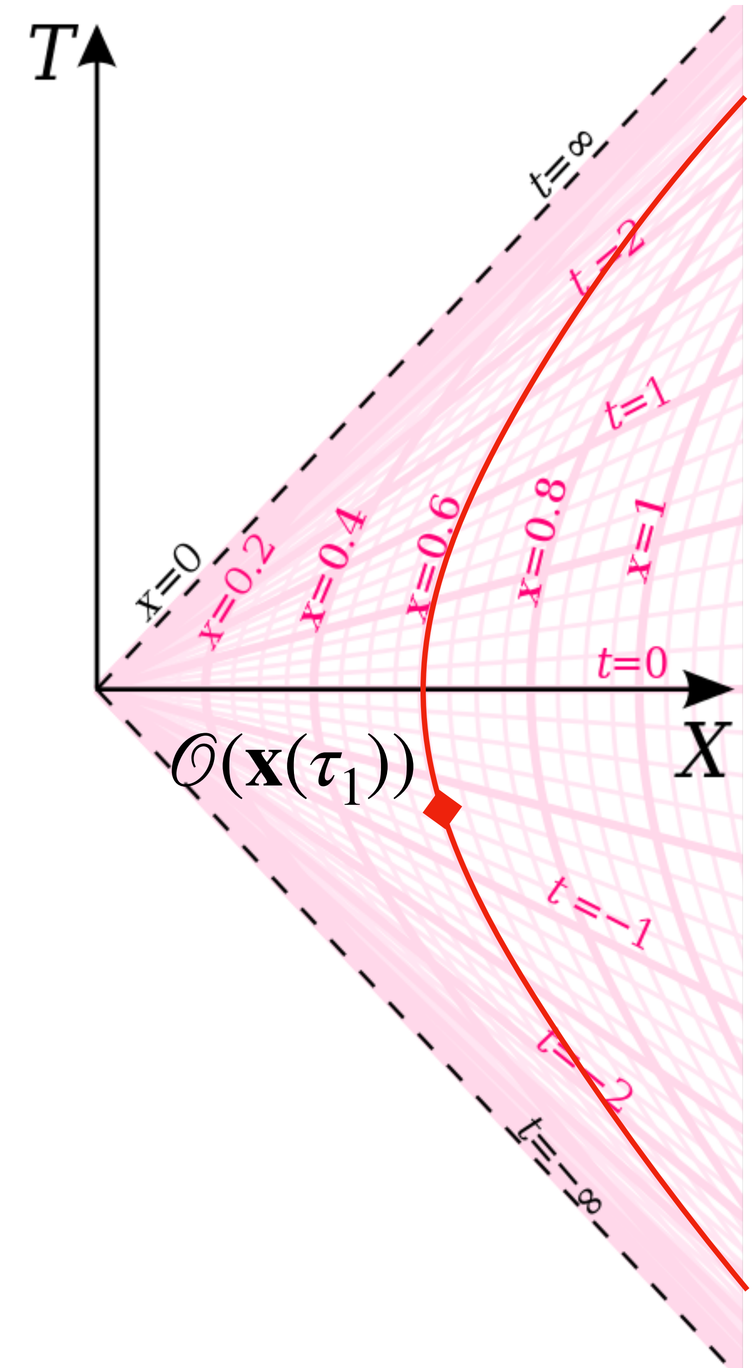

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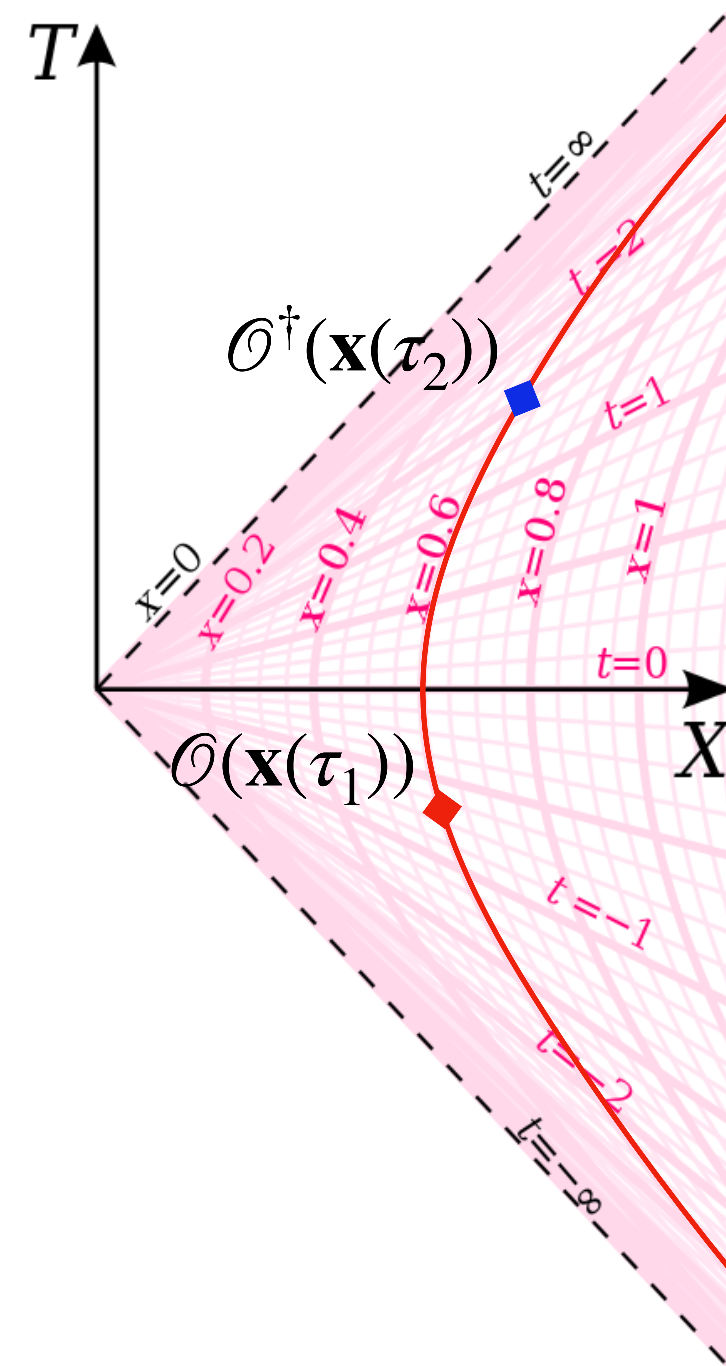

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- Poincare invariance tells us that these functions depend only on the norm and the sign of the time component of $\mathbf{x}(\tau_1) - \mathbf{x}(\tau_2)$.

- So they depend only on $\tau = \tau_1 - \tau_2$.

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- So we only need to consider

- The basic property of real time two-point functions in a thermal ensemble is that there is a holomorphic function on a strip in the complex plane whose boundary values on the two boundaries of the strip are $F(\tau)$ and $G(\tau)$.

- In general, the width of the strip is β, where $\beta = 1/T$ is the inverse temperature.

- Forget it? See page 199

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- We give two derivations of Unruh's result:

    1. starting in real time and deducing the holomorphic properties of the correlation functions;

    2. starting in Euclidean signature and analytically continuing back to real time.

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$$\mathbf{x}(\tau) = R\begin{pmatrix}\sinh(s + i\theta)\\ \cosh(s + i\theta)\end{pmatrix} = R\begin{pmatrix}\cos\theta\sinh s + i\sin\theta\cosh s\\ \cos\theta\cosh s + i\sin\theta\sinh s\end{pmatrix}$$

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- $-\mathbf{Im}(\mathbf{x}(\tau))$ is future timelike $\Rightarrow$ $F(\tau) = \langle \Omega \,|\, \mathcal{O}(\mathbf{x}(\tau))\mathcal{O}^\dagger(\mathbf{x}(0)) \,|\, \Omega \rangle$ is holomorphic

- $-\mathbf{Im}(\mathbf{x}(\tau))$ is past timelike $\Rightarrow$ $G(\tau) = \langle \Omega \,|\, \mathcal{O}^\dagger(\mathbf{x}(0))\mathcal{O}(\mathbf{x}(\tau)) \,|\, \Omega \rangle$ is holomorphic

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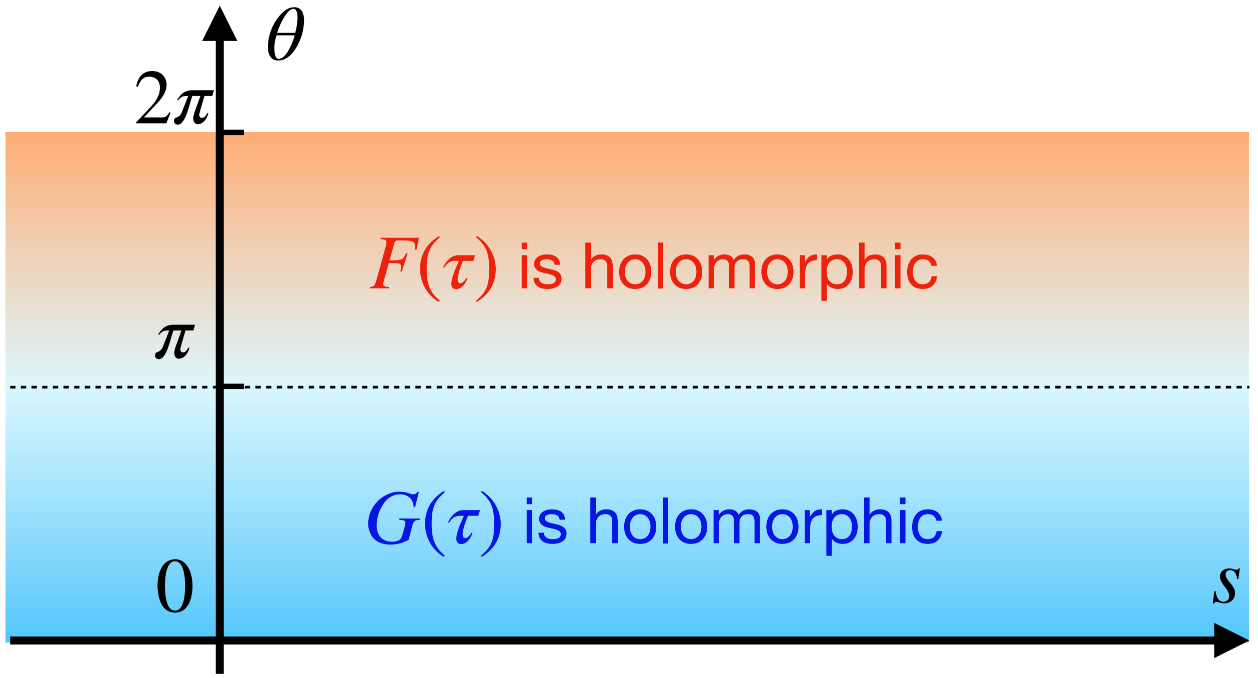

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- Real time method:

- $G(\tau) = \langle \Omega \,|\, \mathscr{O}^\dagger(\mathbf{x}(0))\mathscr{O}(\mathbf{x}(\tau)) \,|\, \Omega \rangle$ is holomorphic in the strip $0 \leqslant \theta \leqslant \pi$, which is $0 \leqslant \mathbf{Im}\tau \leqslant \pi R$;

- $F(\tau) = \langle \Omega \,|\, \mathscr{O}(\mathbf{x}(\tau))\mathscr{O}^\dagger(\mathbf{x}(0)) \,|\, \Omega \rangle$ is holomorphic in the strip $\pi \leqslant \theta \leqslant 2\pi$ (or $-\pi \leqslant \theta \leqslant 0$), which is $\pi R \leqslant \mathbf{Im}\tau \leqslant 2\pi R$ (or $-\pi R \leqslant \mathbf{Im}\tau \leqslant 0$).

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- At $\mathbf{Im}\tau = 0$, $G(\tau) = \langle \Omega \,|\, \mathcal{O}^\dagger(\mathbf{x}(0))\mathcal{O}(\mathbf{x}(\tau)) \,|\, \Omega \rangle$ is simply the original correlation function on the observer's worldline.

- At $\mathbf{Im}\tau = \pi R$, $\mathbf{x}(\tau + i\pi R) = -\,\mathbf{x}(\tau)$ is again real, so the boundary value $G(R(s+i\pi)) = \langle \Omega \,|\, \mathcal{O}^\dagger(\mathbf{x}(0))\mathcal{O}(-\mathbf{x}(Rs)) \,|\, \Omega \rangle$.

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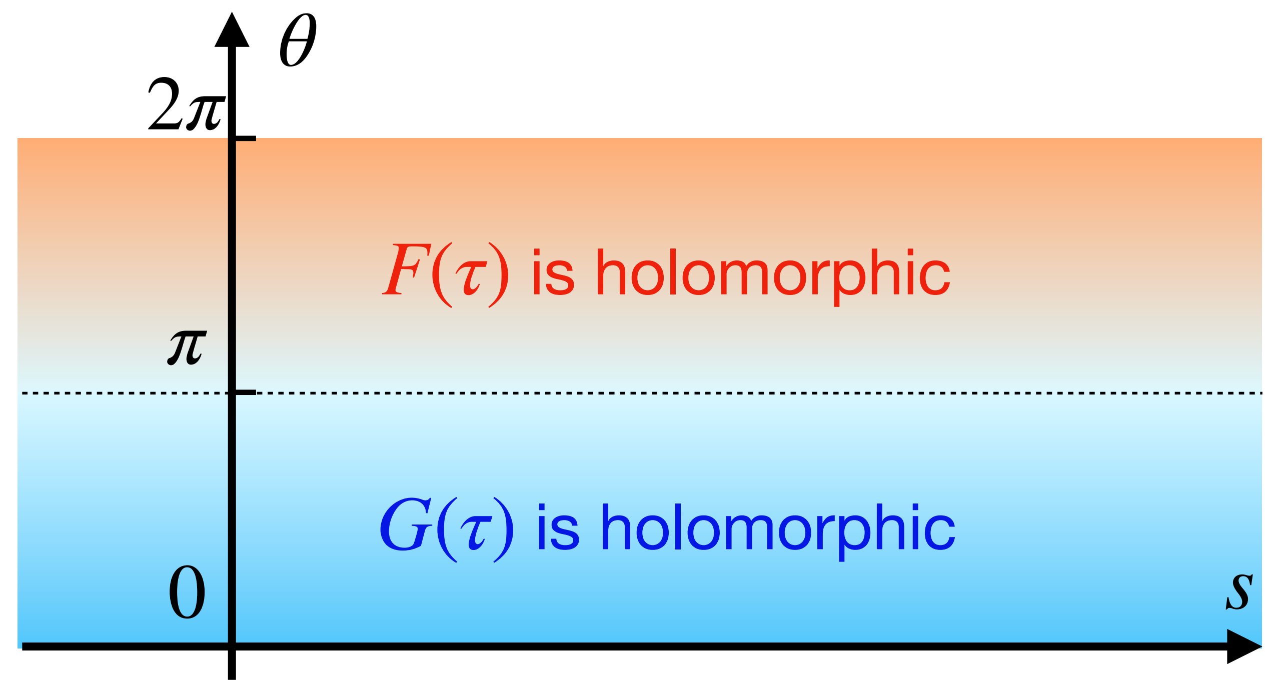
- Real time method:

- In fact, one can define a function $H(\tau)$ which is holomorphic on the combined strip $0 \leqslant \mathbf{Im}\tau \leqslant 2\pi R$ by:

$$H(\tau) = \begin{cases} G(\tau) & 0 \leqslant \mathbf{Im}\tau \leqslant \pi R \\ F(\tau) & \pi R \leqslant \mathbf{Im}\tau \leqslant 2\pi R \end{cases}$$



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- This is the analytic behavior of a real time two-point correlation function in a thermal ensemble with the a strip of width $2\pi R$, so the temperature is $T = 1/(2\pi R)$.

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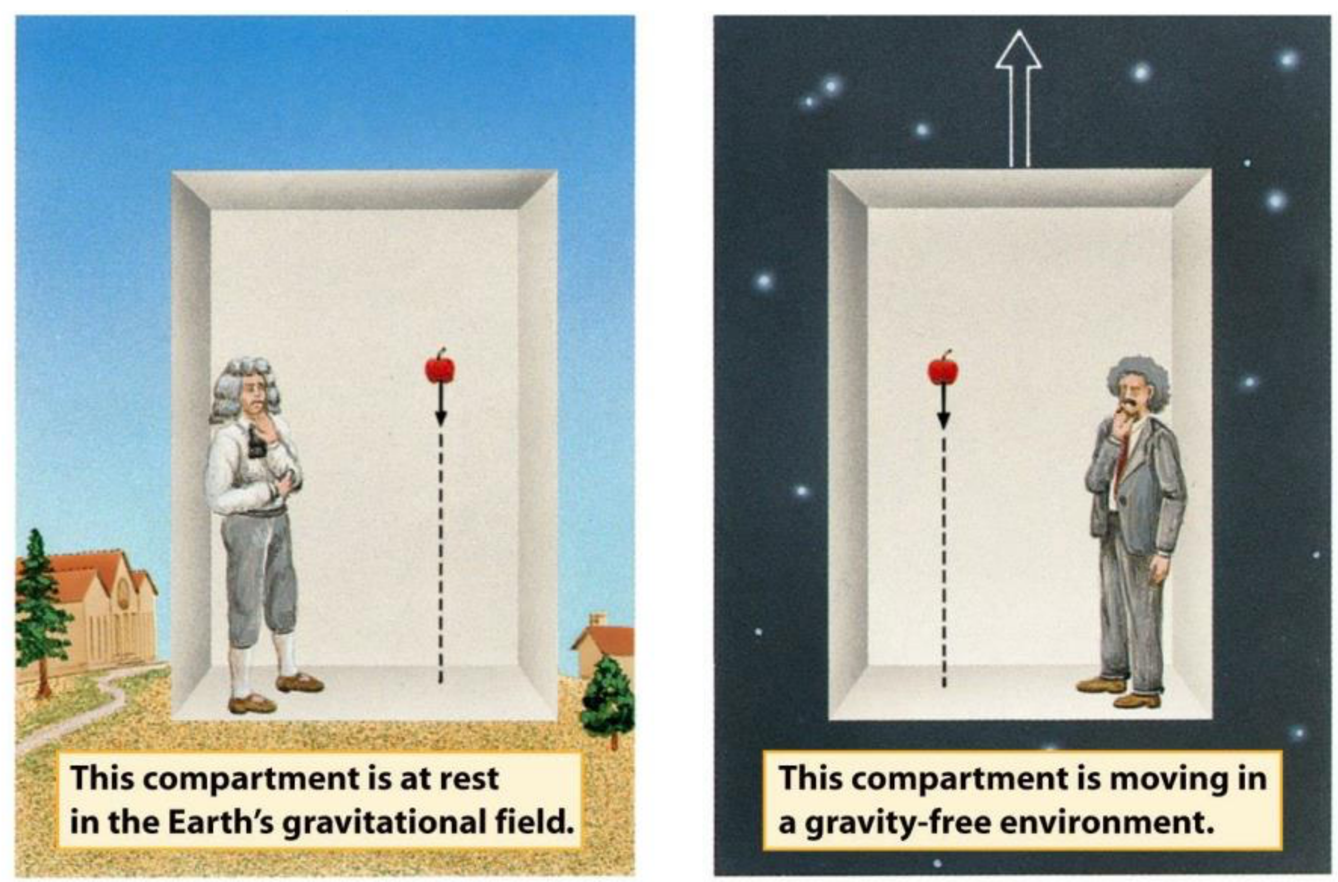
- Unruh's temperature:

- If the equivalence principle of General Relativity is correct, any local measurement can not distinguish a gravitational field from an accelerated frame.



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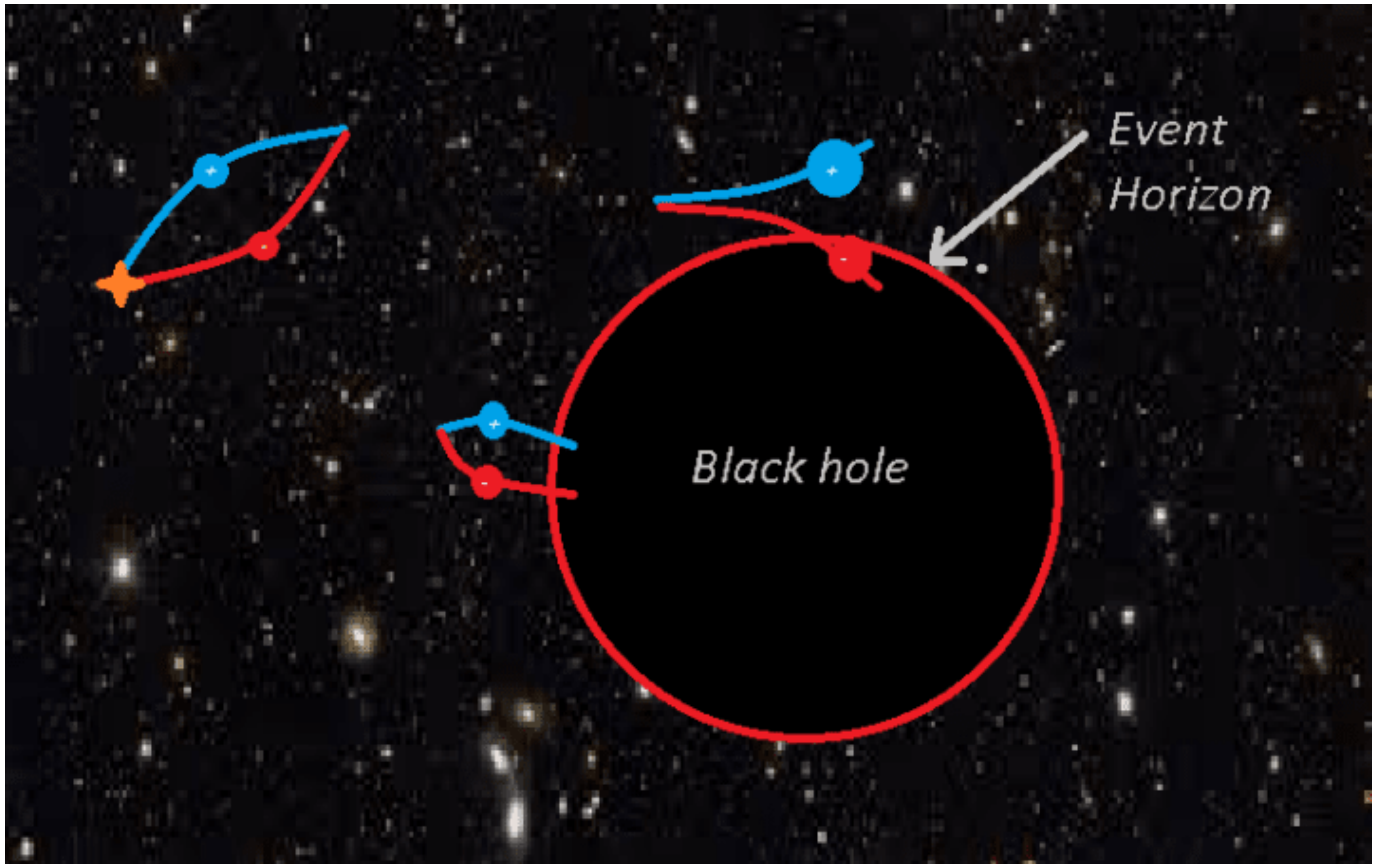
- Unruh's temperature:

- Hawking radiation (non-inertial observers in strong gravitational field)



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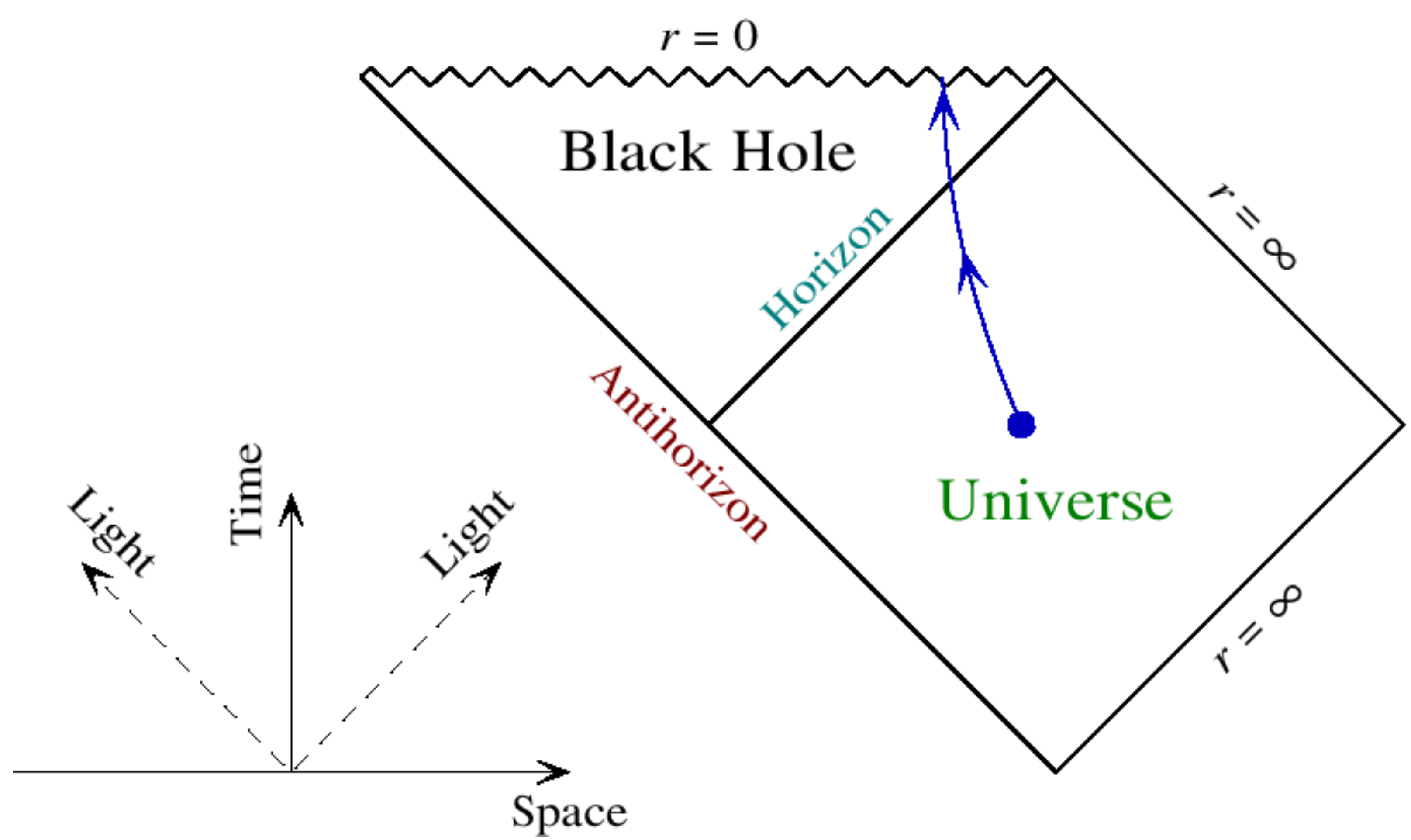

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- Unruh's temperature:
- Hawking radiation (non-inertial observers in strong gravitational field) $\Rightarrow$ what in an accelerating frame?
- An accelerating observer with some style of horizon should measure the "vacuum" as a thermal ensemble.

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- Unruh's temperature:

- The simplest example: massless Hermitian scalar field two-point correlation function.

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$$F(\tau) = \langle \Omega \,|\, \varphi(\mathbf{x}(\tau))\varphi^{\dagger}(\mathbf{x}(0)) \,|\, \Omega \rangle = \langle \Omega \,|\, \varphi(\mathbf{x}(\tau))\varphi(\mathbf{x}(0)) \,|\, \Omega \rangle$$

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$$= \int \frac{d^{D-1}\mathbf{p}\, d^{D-1}\mathbf{q}}{(2\pi)^{2(D-1)}\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \langle \Omega \,|\, a_{\mathbf{p}} a^{\dagger}_{\mathbf{q}} e^{-ip\cdot\mathbf{x}(\tau)+iq\cdot\mathbf{x}(0)} \,|\, \Omega \rangle$$

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- The integral of the angular coordinates are

$$\int d^{D-2}\Omega_p = \int_0^\pi \sin^{D-3}\varphi_1 d\varphi_1 \int_0^\pi \sin^{D-4}\varphi_2 d\varphi_2 \cdots \int_0^{2\pi} \varphi_{D-2} d\varphi_{D-2}$$

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$$F(\tau) = \frac{1}{2(2\pi)^{D-1}} \int_0^{\infty} p^{D-3} dp \int_0^{\pi} \sin^{D-3} \varphi_1 d\varphi_1 \int_0^{\pi} \sin^{D-4} \varphi_2 d\varphi_2 \cdots \int_0^{2\pi} \varphi_{D-2} d\varphi_{D-2} e^{-ip(\Delta t - \Delta x \cos \varphi_1)}$$

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$$= \frac{\pi^{-2+D/2}}{2(2\pi)^{D-2}} \prod_{n=1}^{D-4} \frac{\Gamma((1+n)/2)}{\Gamma(1+n/2)} \int_0^\pi (i(\Delta t - \Delta x \cos \varphi_1))^{2-D} \Gamma(D-2) \sin^{D-3} \varphi_1 d\varphi_1$$

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- The simplest example: massless Hermitian scalar field two-point correlation function.

$$
\begin{aligned}
F(\tau) &= \frac{1}{2(2\pi)^{D-1}} \int_0^\infty p^{D-3} dp \int_0^\pi \sin^{D-3}\varphi_1 d\varphi_1 \int_0^\pi \sin^{D-4}\varphi_2 d\varphi_2 \cdots \int_0^{2\pi} \varphi_{D-2} d\varphi_{D-2} e^{-ip(\Delta t - \Delta x \cos\varphi_1)} \\
&= \frac{2\pi}{2(2\pi)^{D-1}} \prod_{n=1}^{D-4} \frac{\sqrt{\pi}\Gamma((1+n)/2)}{\Gamma(1+n/2)} \int_0^\infty p^{D-3} dp \int_0^\pi e^{-ip(\Delta t - \Delta x \cos\varphi_1)} \sin^{D-3}\varphi_1 d\varphi_1 \\
&= \frac{\pi^{-2+D/2}}{2(2\pi)^{D-2}} \prod_{n=1}^{D-4} \frac{\Gamma((1+n)/2)}{\Gamma(1+n/2)} \int_0^\pi (i(\Delta t - \Delta x \cos\varphi_1))^{2-D} \Gamma(D-2) \sin^{D-3}\varphi_1 d\varphi_1 \\
&= \frac{\pi^{(D-3)/2}}{2(2\pi i)^{D-2}(\Delta t^2 - \Delta x^2)^{D/2-1}} \prod_{n=1}^{D-3} \frac{\Gamma((1+n)/2)}{\Gamma(1+n/2)} = \frac{\pi^{(D-3)/2}}{2(2\pi i)^{D-2}\Gamma((D-1)/2)(\Delta t^2 - \Delta x^2)^{D/2-1}}
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- The result requires $\mathbf{Im}(\Delta t \pm \Delta x) < 0$ and $D > 2$

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$$\Pi(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} \langle \Omega | \varphi(t)\varphi(0) | \Omega \rangle dt = \int_{-\infty}^{+\infty} e^{-i\omega\tau} F(\tau) d\tau$$

A FUNDAMENTAL EXAMPLE

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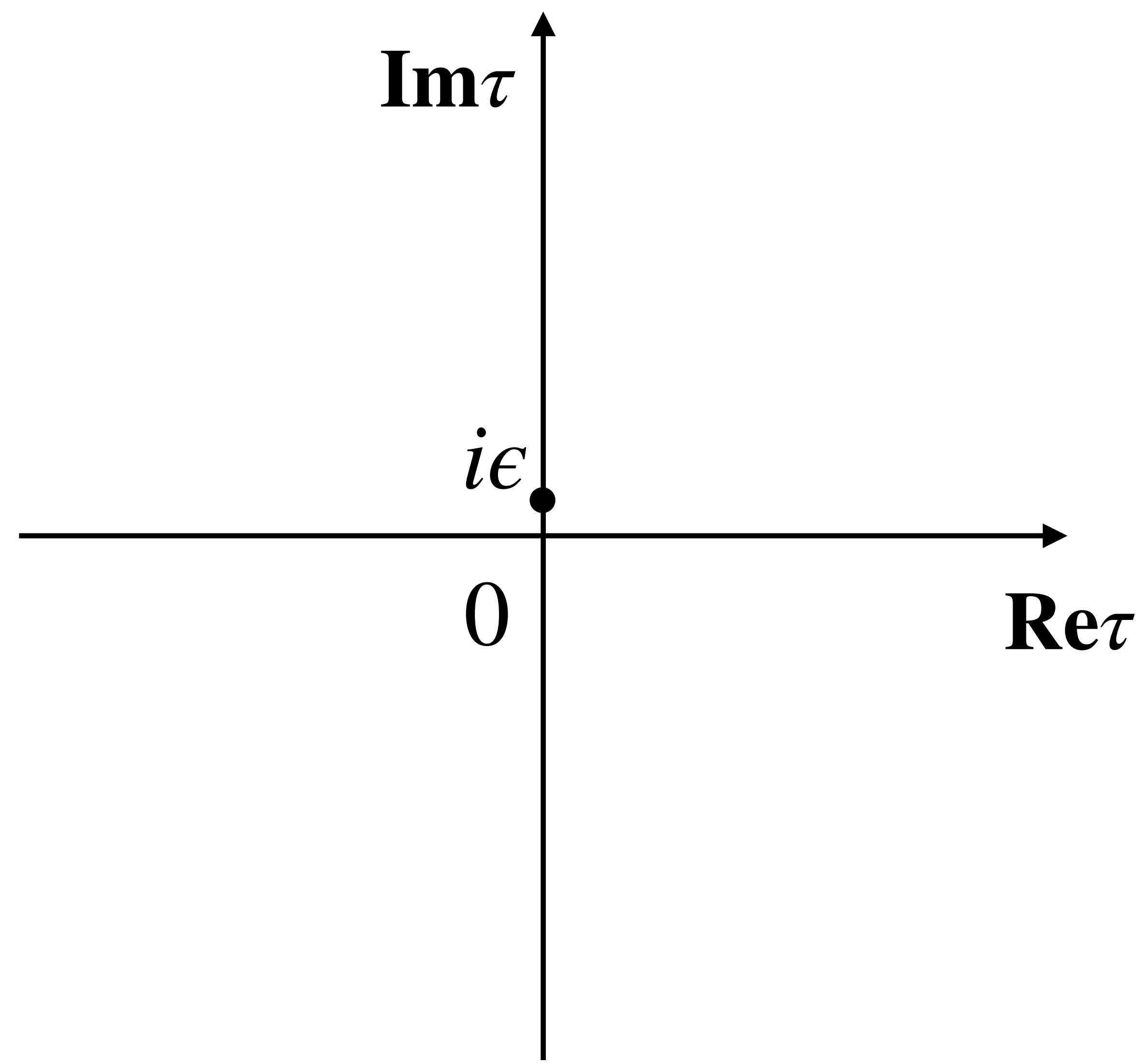

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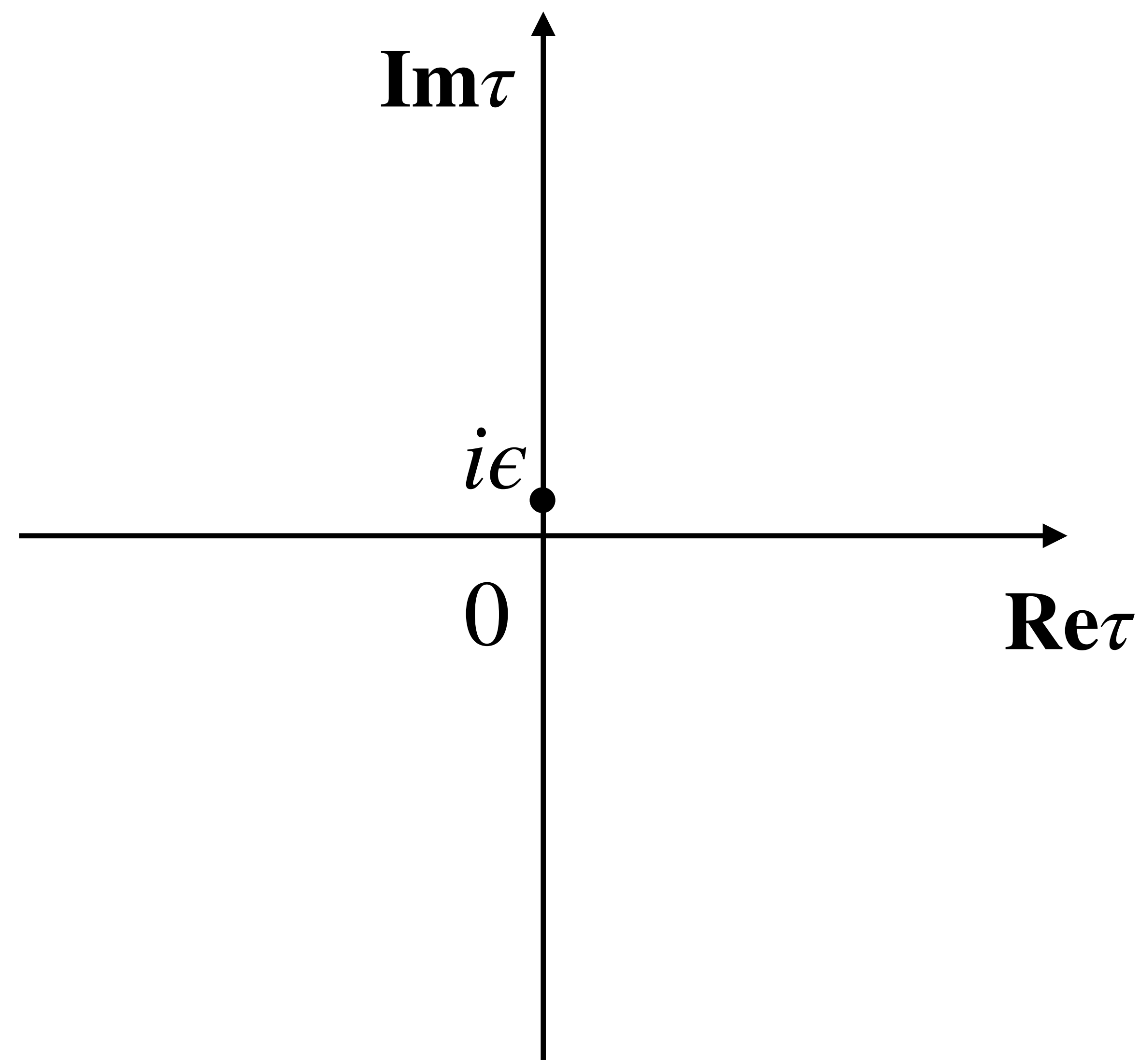

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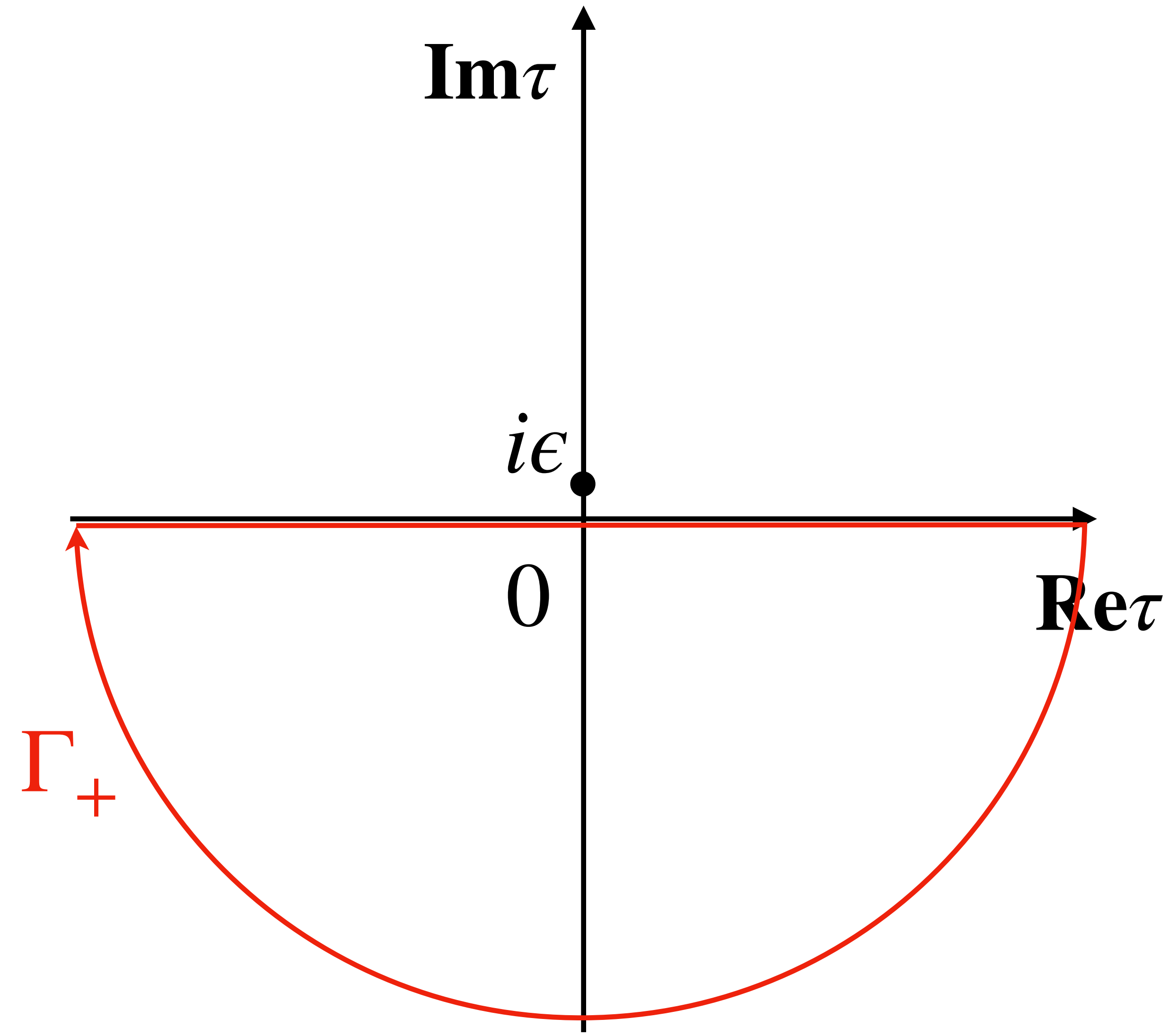

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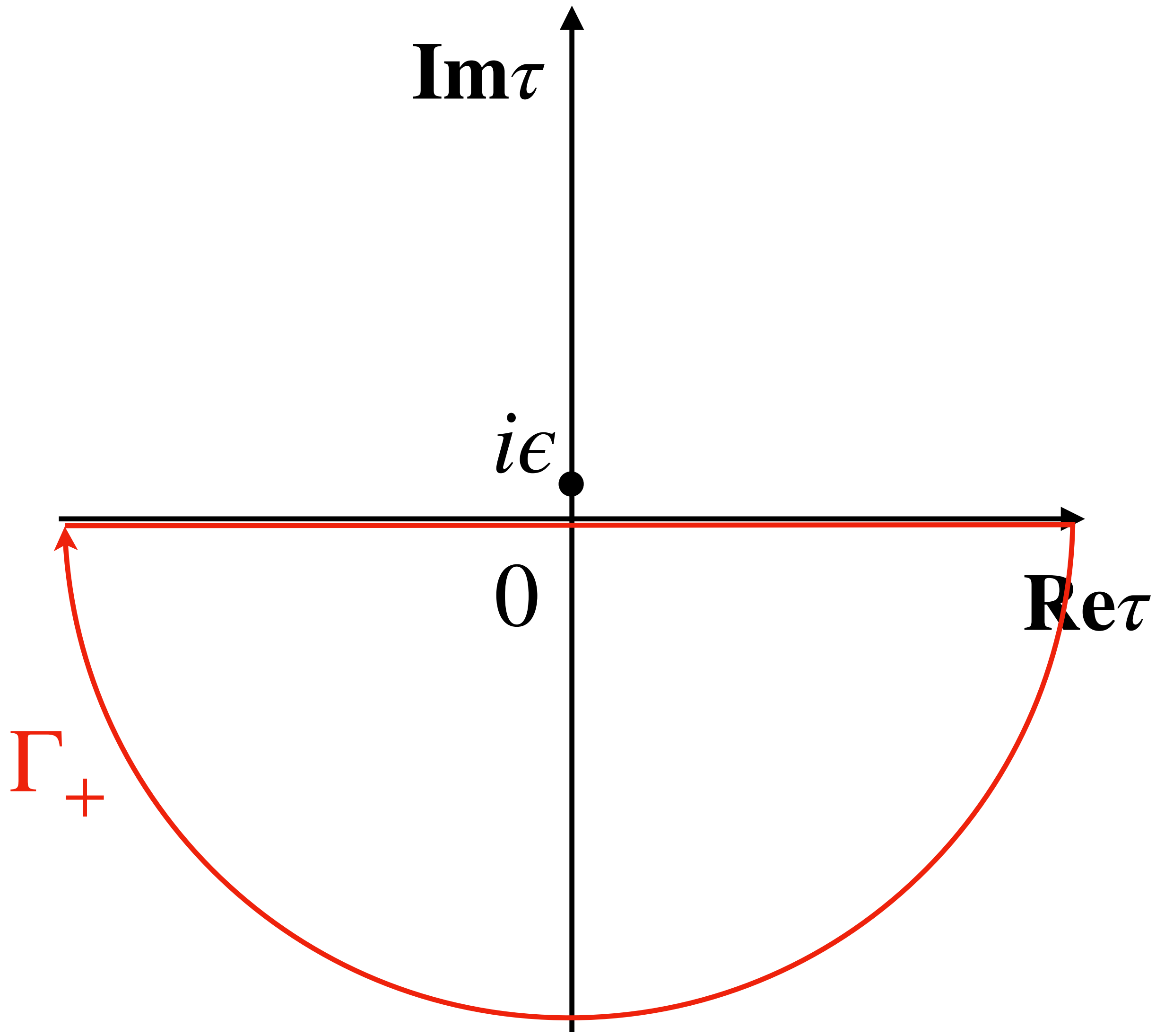

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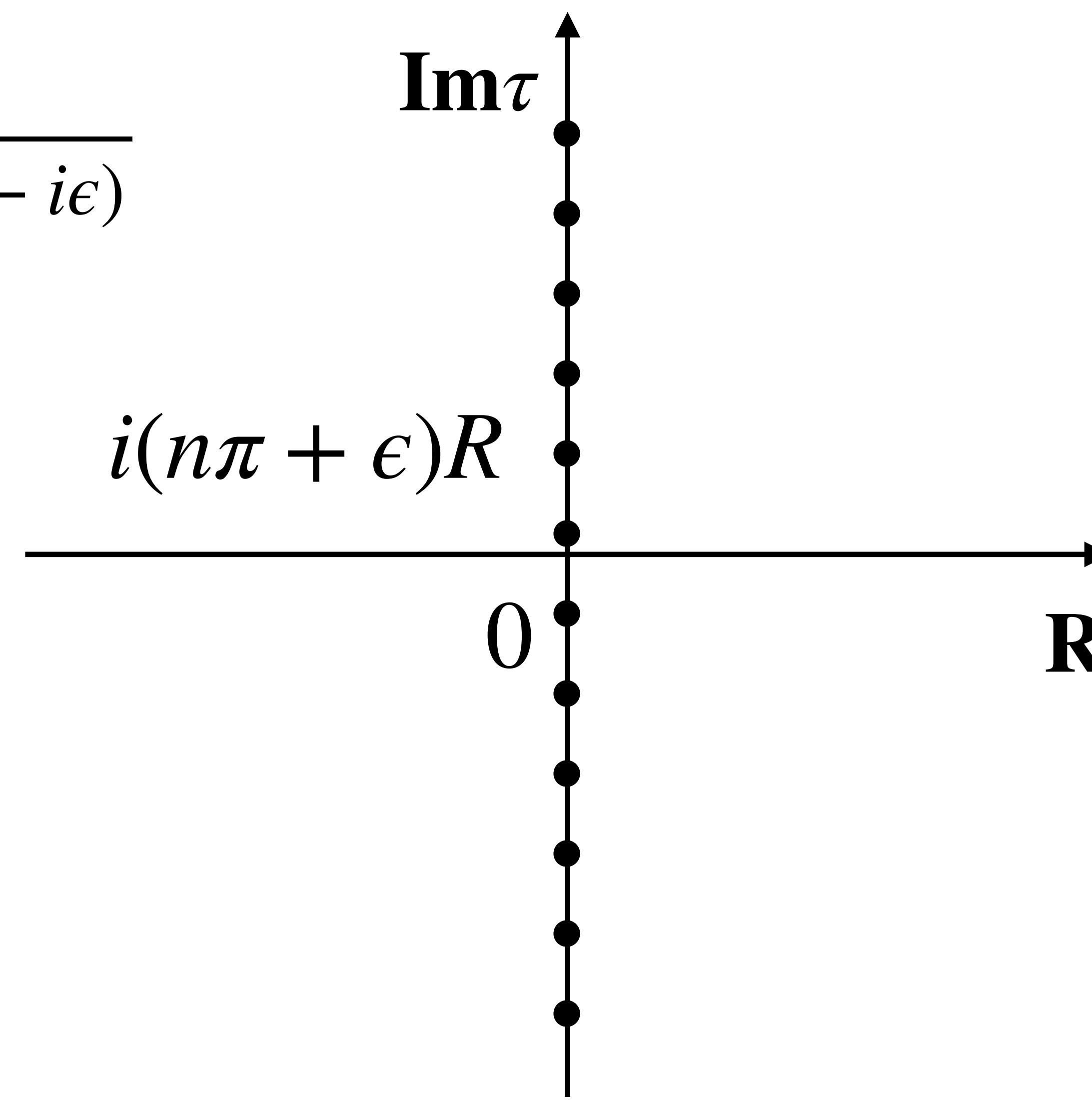

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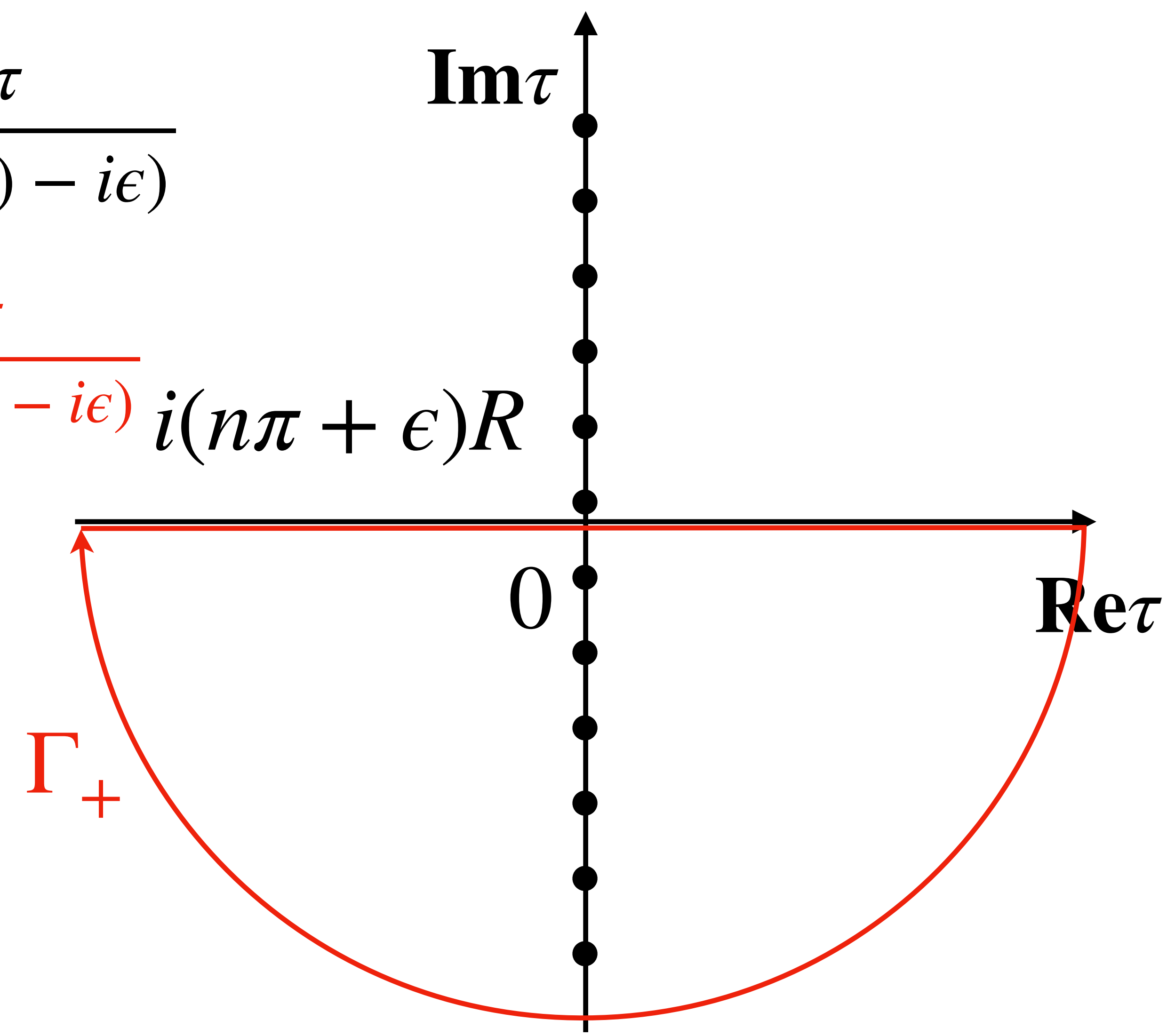

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&= -\frac{i}{2\pi R^2} \sum_{n=1}^{\infty} (-4i) R^2 \omega \left(e^{-2\pi R\omega} \right)^n \\
&= \frac{2\omega}{\pi} \frac{1}{e^{2\pi R\omega} - 1}
\end{aligned}$$

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Extremely Low!!!

A FUNDAMENTAL EXAMPLE

IV. An accelerating observer

- Euclidean method: (more transparent)

- The Euclidean version ($t_E = it$) of the worldline of the uniformly accelerated observer is:

$$\begin{pmatrix} t_E(\theta) \\ x(\theta) \end{pmatrix} = R \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

- The method is quite straightforward.

- In this slides, we will ignore this method which is given shortly in Witten's paper.


To Be Continued...