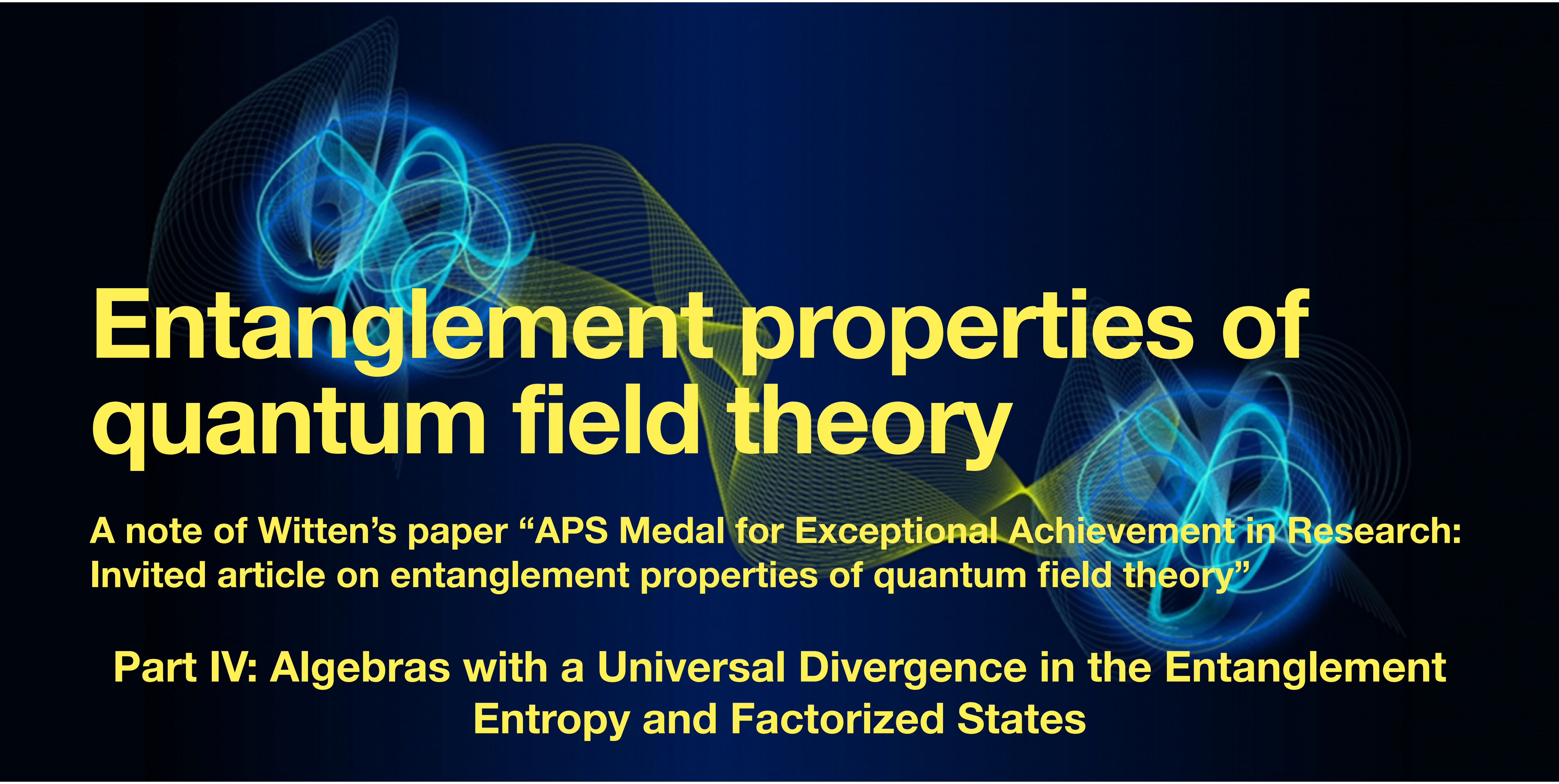


Entanglement properties of quantum field theory

A note of Witten's paper "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory"

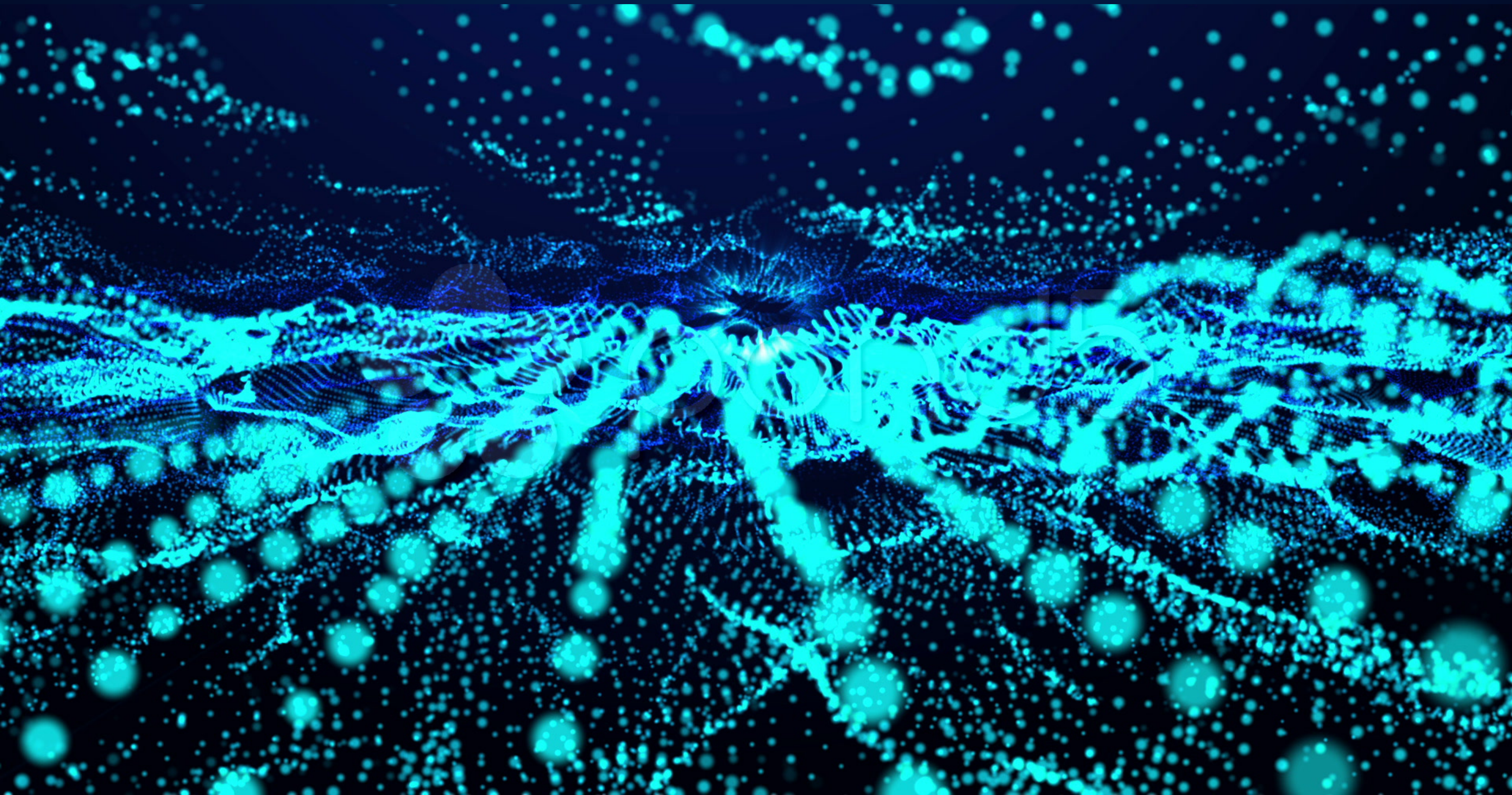
Part IV: Algebras with a Universal Divergence in the Entanglement Entropy and Factorized States

Hao Zhang
Theoretical Physics Division, Institute of High Energy Physics, Chinese Academy of Sciences

A Review

- The Reeh-Schlieder Theorem
- The Modular Operator and Relative Entropy
- Finite-dimensional Quantum Systems and Some Lessons
- A Fundamental Example

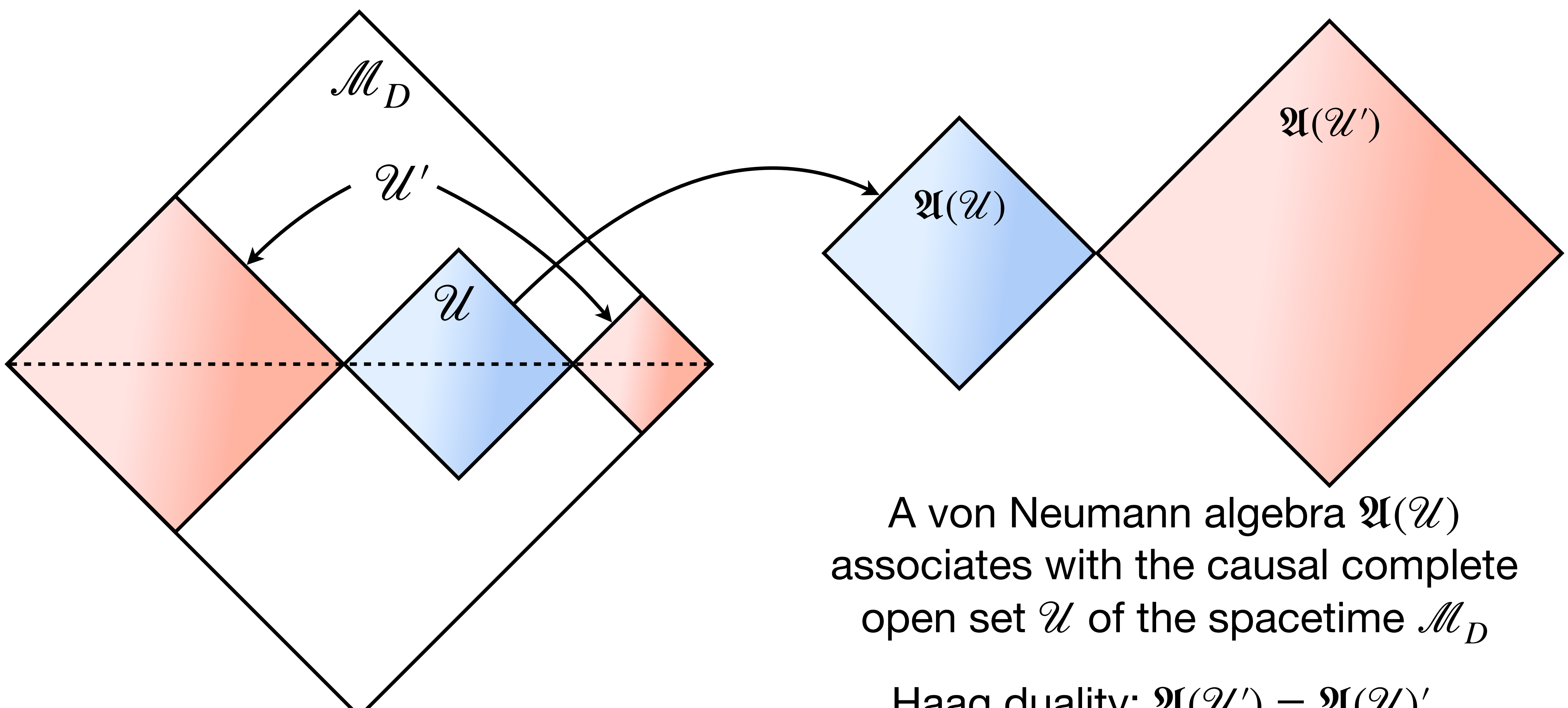
ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY



ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

I. The problem

- Let $\mathscr{U}$ be an open set in Minkowski spacetime $\mathscr{M}_D$, it has a local algebra $\mathfrak{A} = \mathfrak{A}(\mathscr{U})$ with commutant $\mathfrak{A}'$ (which, if Haag duality holds, is $\mathfrak{A}(\mathscr{U}')$ for some other open set $\mathscr{U}'$)



A von Neumann algebra $\mathfrak{A}(\mathscr{U})$ associates with the causal complete open set $\mathscr{U}$ of the spacetime $\mathscr{M}_D$

Haag duality: $\mathfrak{A}(\mathscr{U}') = \mathfrak{A}(\mathscr{U})'$

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- $\mathfrak{A}$ and $\mathfrak{A}'$ are von Neumann algebras of bounded operators which act on the Hilbert space $\mathscr{H}$ of the theory in question with the vacuum state $|\Omega\rangle$ as a cyclic separating vector.

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- $\mathfrak{A}$ and $\mathfrak{A}'$ are von Neumann algebras of bounded operators which act on the Hilbert space $\mathscr{H}$ of the theory in question with the vacuum state $|\Omega\rangle$ as a cyclic separating vector.

- For a finite-dimensional quantum system (quantum mechanics), the existence of such a cyclic separating vector would imply a factorization $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, with $\mathfrak{A}$ acting on one factor and $\mathfrak{A}'$ acting on the other.

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- Such a factorization cannot exist in quantum field theory, for it would imply the existence of tensor product states $|\psi\rangle \otimes |\chi\rangle$ with no entanglement between $\mathscr{U}$ and $\mathscr{U}'$.

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- $\mathfrak{A}$ and $\mathfrak{A}'$ are von Neumann algebras of bounded operators which act on the Hilbert space $\mathscr{H}$ of the theory in question with the vacuum state $|\Omega\rangle$ as a cyclic separating vector.

- Such a factorization cannot exist in quantum field theory, for it would imply the existence of tensor product states $|\psi\rangle \otimes |\chi\rangle$ with no entanglement between $\mathscr{U}$ and $\mathscr{U}'$.

- Instead, in quantum field theory, there is a universal ultraviolet divergence in the entanglement entropy.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

I. The problem

- The essence of the matter is that in quantum field theory, the divergence in the entanglement entropy is not a property of the states but of the algebras $\mathfrak{A}$ and $\mathfrak{A}'$.

- It means that the divergence is an essential property of the algebras but not of some specific representations of the algebra.

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- Mathematically, these algebras are not the familiar type I von Neumann algebras which can act irreducibly (have irreducible representation) in a Hilbert space.

- Instead they are more exotic algebras with property that the structure of the algebra has the divergence in the entanglement entropy built in.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

I. The problem

- We will explain barely enough about von Neumann algebras to indicate how that comes about in this section. (Murray and von Neumann, 1936)

Neumann János Lajos
(1903/12/28-1957/02/08)

Francis Joseph Murray
(1911/02/03-1996/03/15)

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

I. The problem

- Before going to the next section, we first give a mathematically rigorous definition of von Neumann algebra as a supplementary material.

- Do not like C*-algebra, because the definition of the weak operator topology of the von Neumann algebra depends on the Hilbert space, people usually use a concrete definition of von Neumann algebra.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

I. The problem

- A **von Neumann algebra** on Hilbert space $\mathscr{H}$ is a subalgebra $\mathfrak{M}$ of the bounded operator $\mathscr{B}(\mathscr{H})$ which is closed under involution (the *-operation) and $\mathfrak{M}'' = \mathfrak{M}$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

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- We will explain barely enough about von Neumann algebras to indicate how that comes about in this section.

- The discussion will be limited on the fundamental block of the von Neumann algebra — **factor**.

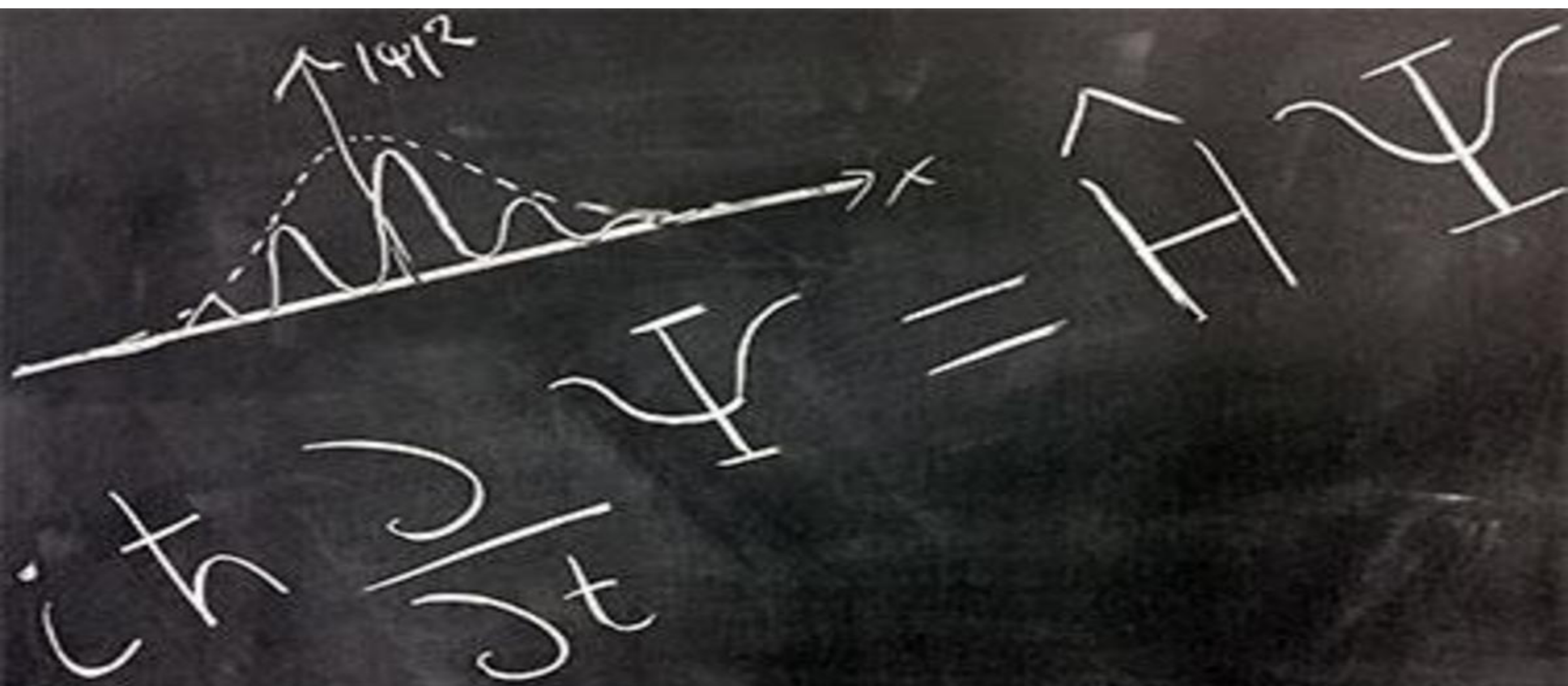
- A von Neumann algebra is called a factor, if it has a trivial center.

$$\mathfrak{A} \text{ is a factor } \Leftrightarrow \mathfrak{A} \cap \mathfrak{A}' = \mathbb{C} \cdot \mathbf{1}$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

II. Algebras of type I

- A type $\mathbf{I}$ von Neumann algebra $\mathfrak{A}$ can act irreducibly by bounded operators on a Hilbert space $\mathscr{K}$.



ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

II. Algebras of type I

- A type $\mathbf{I}$ von Neumann algebra $\mathfrak{A}$ can act irreducibly by bounded operators on a Hilbert space $\mathscr{K}$.

- Because we require $\mathfrak{A}$ to be a factor, it actually consists of all bounded operators on $\mathscr{K}$.

- A von Neumann algebra (with trivial center) acting irreducibly on a (at most separated) Hilbert space is always of one of two types

  1. Type $\mathbf{I}_d$: $\dim \mathscr{K} = d < \infty$;

  2. Type $\mathbf{I}_\infty$: $\dim \mathscr{K} = \aleph_1$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

II. Algebras of type I

- **Trace**: a trace on a von Neumann algebra is a linear function $\mathbf{Tr}: \ \mathbf{a} \in \mathfrak{A} \ \rightarrow \mathbf{Tr}(\mathbf{a}) \in \mathbb{C}$ that satisfies $\mathbf{Tr}(\mathbf{ab}) = \mathbf{Tr}(\mathbf{ba})$ and $\mathbf{Tr}(\mathbf{a}^{\dagger}\mathbf{a}) > 0$ for $\mathbf{a} \neq 0$.

- It is obviously that any algebra of type $\mathbf{I}_d$ has a trace.

- For type $\mathbf{I}_{\infty}$, one can also define a trace except that it can not be defined on the whole algebra.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- We will give a quick description of the algebras of type $\mathbf{II}$.

- It can be constructed as follows from a countably infinite set of maximally entangled qubit pairs.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- Let V be a vector space consisting of 2×2 complex matrices with Hilbert space structure defined by $(v, w) = \mathbf{Tr}(v^\dagger w)$.

- A bipartite system

$$\left.\begin{array}{l} |\Psi_A\rangle = a_1 | \uparrow_A \rangle + a_2 | \downarrow_A \rangle \\ |\Psi_B\rangle = b_1 | \uparrow_B \rangle + b_2 | \downarrow_B \rangle \end{array}\right\} \to |\Psi_{AB}\rangle = \begin{pmatrix} | \uparrow_A \rangle & | \downarrow_A \rangle \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} | \uparrow_B \rangle \\ | \downarrow_B \rangle \end{pmatrix}$$

$$= a_{11} | \uparrow_A \uparrow_B \rangle + a_{12} | \uparrow_A \downarrow_B \rangle + a_{21} | \downarrow_A \uparrow_B \rangle + a_{22} | \downarrow_A \downarrow_B \rangle$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

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$$= a_{11}|\uparrow_A \uparrow_B\rangle + a_{12}|\uparrow_A \downarrow_B\rangle + a_{21}|\downarrow_A \uparrow_B\rangle + a_{22}|\downarrow_A \downarrow_B\rangle$$

- The algebra M_A (M_B) of the operators of subsystem A (B) is the algebra of 2×2 complex matrices I_2.

- The operator $a_A \in M_A$ ($a_B \in M_B$) acts on V on the left (right) by $v \rightarrow a_A v$ ($v \rightarrow v a_B^T$).

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- The algebra M_A (M_B) of the operators of subsystem A (B) is the algebra of 2×2 complex matrices $M_2(\mathbb{C})$.

- The operator $a_A \in M_A$ ($a_B \in M_B$) acts on V on the left (right) by $v \to a_A v$ ($v \to v a_B^T$).

- It is obviously that M_A and M_B are commutants.

$$\hat{a}_A \hat{a}_B |\psi\rangle = \begin{pmatrix} |\uparrow_A\rangle & |\downarrow_A\rangle \end{pmatrix} a_A \left[v_\psi a_B^T \begin{pmatrix} |\uparrow_B\rangle \\ |\downarrow_B\rangle \end{pmatrix} \right]$$

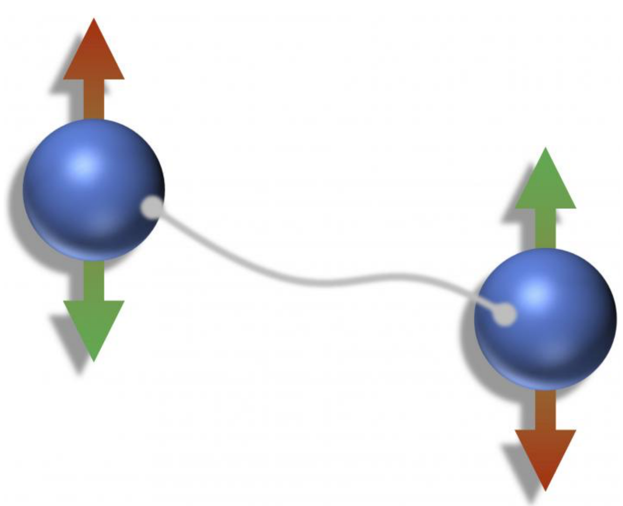
$$= \left[\begin{pmatrix} |\uparrow_A\rangle & |\downarrow_A\rangle \end{pmatrix} a_A v_\psi \right] a_B^T \begin{pmatrix} |\uparrow_B\rangle \\ |\downarrow_B\rangle \end{pmatrix} = \hat{a}_B \hat{a}_A |\psi\rangle$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- Now consider a countably infinite set of copies of this construction.

- For $k \geqslant 1$, let $V^{[k]}$ be a space of 2×2 matrices acted on on the left by $M_A^{[k]}$ and on the right by $M_B^{[k]}$.



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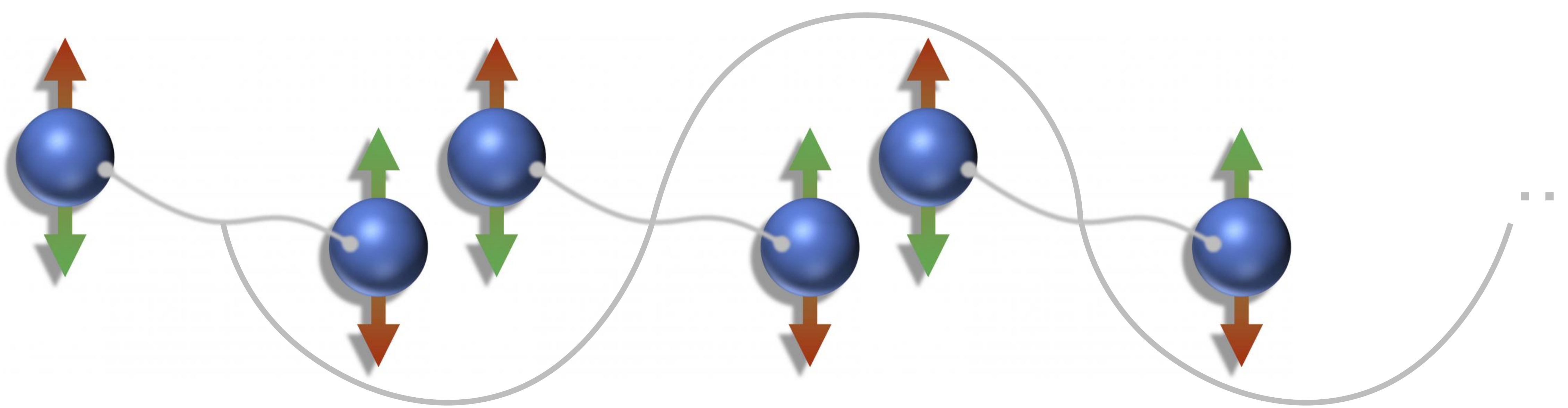

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- Roughly speaking, we want to consider the infinite tensor product $V^{[1]} \otimes V^{[2]} \otimes \cdots \otimes V^{[k]} \otimes \cdots$. The dimension of such tensor product space is $\aleph_1^{\aleph_1}$, which is uncountable.

- To get a Hilbert space of countably infinite dimension, we define a space $\mathscr{H}_0$ that consists of tensor products $v_1 \otimes v_2 \otimes \cdots \otimes v_k \otimes \cdots \in V^{[1]} \otimes V^{[2]} \otimes \cdots \otimes V^{[k]} \otimes \cdots$ such that all but finitely many of the v_k are equal to $\mathbf{1}'_{2\times 2} = 2^{-1/2}\mathbf{1}_{2\times 2}$.

- The inner product is defined by $(v, w) = \prod_{i=1}^{\infty} \mathbf{Tr} v_i^\dagger w_i = \prod_{i=1}^{n} \mathbf{Tr} v_i^\dagger w_i$.

- One completes it to get a Hilbert space $\mathscr{H}$, which is called a restricted tensor product of the $V^{[k]}$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- We also want to define an algebra $\mathfrak{A}$ as an infinite tensor product $M_A^{[1]} \otimes M_A^{[2]} \otimes \cdots \otimes M_A^{[k]} \otimes \cdots$.

- A general element is $a_{\mathfrak{A}} = a_A^{[1]} \otimes a_A^{[2]} \otimes \cdots \otimes a_A^{[k]} \otimes \cdots$.

- However, it would not preserve the condition that all but finitely many of the v_k are equal to $\mathbf{1}'_{2\times 2}$!

- So we have to first define the algebra $\mathfrak{A}_0$ that consists of elements $a_{\mathfrak{A}} = a_A^{[1]} \otimes a_A^{[2]} \otimes \cdots \otimes a_A^{[k]} \otimes \cdots$ such that all but finitely many of the $a_A^{[k]}$ are equal to $\mathbf{1}_{2\times 2}$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- The algebra $\mathfrak{A}_0$ acts on the left on $\mathscr{H}$.

- One needs to add the limit point to make it closed under the weak operator topology.

- A sequence $a_{\mathfrak{A}}^{(k)} \in \mathfrak{A}_0$ is (weak) convergence if $\lim_{k\to\infty} a_{\mathfrak{A}}^{(k)}\chi$ exists for all $\chi \in \mathscr{H}$; if so, we define an operator $a_{\mathfrak{A}} : \mathscr{H} \to \mathscr{H}$ by $a_{\mathfrak{A}}\chi = \lim_{k\to\infty} a_{\mathfrak{A}}^{(k)}\chi$, and we define $\mathfrak{A}$ to include all such limits.

- This definition ensures that for $a_{\mathfrak{A}} \in \mathfrak{A}$, $\chi \in \mathscr{H}$, $a_{\mathfrak{A}}\chi$ is a continuous function of $a_{\mathfrak{A}}$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- Note that the definition of the algebra depends on a knowledge of the Hilbert space.

- The commutant of $\mathfrak{A}$ is an isomorphic algebra $\mathfrak{B}$ which is defined in just the same way as a subalgebra of $M_B^{[1]} \otimes M_B^{[2]} \otimes \cdots \otimes M_B^{[k]} \otimes \cdots$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- It is obviously that the vector $\Psi = \mathbf{1}'_{2\times 2} \otimes \mathbf{1}'_{2\times 2} \otimes \cdots \otimes \mathbf{1}'_{2\times 2} \otimes \cdots \in \mathscr{H}$ is cyclic separating for both $\mathfrak{A}$ and $\mathfrak{B}$.

- A natural linear function on $\mathfrak{A}$ is defined by $F(a_{\mathfrak{A}}) = \langle \Psi \,|\, a_{\mathfrak{A}} \,|\, \Psi \rangle$.

- Because Ψ is separating for $\mathfrak{A}$, any nonzero $a_{\mathfrak{A}} \in \mathfrak{A}$ satisfies $a_{\mathfrak{A}} \,|\, \Psi \rangle \neq 0$ and hence $F(a_{\mathfrak{A}}^{\dagger} a_{\mathfrak{A}}) > 0$.

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$$F(a_{\mathfrak{A}} b_{\mathfrak{A}}) = \langle \Psi \,|\, a_{\mathfrak{A}} b_{\mathfrak{A}} \,|\, \Psi \rangle = 2^{-k} \prod_{i=1}^{k<\infty} \mathbf{Tr}(a_{\mathfrak{A}}^{[i]} b_{\mathfrak{A}}^{[i]}) = 2^{-k} \prod_{i=1}^{k<\infty} \mathbf{Tr}(b_{\mathfrak{A}}^{[i]} a_{\mathfrak{A}}^{[i]})$$

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$$
\begin{aligned}
F(a_{\mathfrak{A}} b_{\mathfrak{A}}) = \langle \Psi \,|\, a_{\mathfrak{A}} b_{\mathfrak{A}} \,|\, \Psi \rangle &= 2^{-k} \prod_{i=1}^{k<\infty} \mathbf{Tr}(a^{[i]}_{\mathfrak{A}} b^{[i]}_{\mathfrak{A}}) = 2^{-k} \prod_{i=1}^{k<\infty} \mathbf{Tr}(b^{[i]}_{\mathfrak{A}} a^{[i]}_{\mathfrak{A}}) \\
&= \langle \Psi \,|\, b_{\mathfrak{A}} a_{\mathfrak{A}} \,|\, \Psi \rangle = F(b_{\mathfrak{A}} a_{\mathfrak{A}})
\end{aligned}
$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- Since elements $a_{\mathfrak{A}}, b_{\mathfrak{A}}$ of the form considered are dense in $\mathfrak{A}$, $F(a_{\mathfrak{A}} b_{\mathfrak{A}}) = F(b_{\mathfrak{A}} a_{\mathfrak{A}})$ exists for any $a_{\mathfrak{A}}, b_{\mathfrak{A}} \in \mathfrak{A}$.

- So $F(a_{\mathfrak{A}}) = \langle \Psi \,|\, a_{\mathfrak{A}} \,|\, \Psi \rangle$ defines a trace on $\mathfrak{A}$, we denote it as $\mathbf{Tr}(a_{\mathfrak{A}})$.

- Because Ψ is separating for $\mathfrak{A}$, any nonzero $a_{\mathfrak{A}} \in \mathfrak{A}$ satisfies $a_{\mathfrak{A}} \,|\, \Psi \rangle \neq 0$ and hence $F(a_{\mathfrak{A}}^\dagger a_{\mathfrak{A}}) > 0$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- In the case of a type $\mathbf{I}_\infty$ algebra, one can define a trace on a subalgebra but the trace of the identity element is infinite.

- By contrast, a hyperfinite type $\mathbf{II}_1$ algebra has a trace that is defined on the whole algebra, and which we have normalized so that $\mathbf{Tr}(\mathbf{1}_{\mathfrak{A}}) = 1$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- The entanglement entropy in the state Ψ.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

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$$\mathcal{S}_{\Psi} = -\,\mathbf{Tr}_{\mathfrak{A}}\left(\rho_{\mathfrak{A}} \log \rho_{\mathfrak{A}}\right) = -\sum_{k=1}^{\infty} \mathbf{Tr}_{\mathfrak{A}[k]}\left(\rho_{\mathfrak{A}[k]} \log \rho_{\mathfrak{A}[k]}\right)$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

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ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

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- The divergence is due to the fact that each factor of $\mathbf{1}'_{2\times 2}$ represents a perfectly entangled qubit pair shared between $\mathfrak{A}$ and $\mathfrak{B}$.

- Replacing Ψ by another state in $\mathscr{H}$ will only change the entanglement entropy by a finite or at least less divergent amount. Because there are always infinite $\mathbf{1}'_{2\times 2}$ factors in a state by definition.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

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- So the leading divergence in the entanglement entropy in a hyperfinite type $\mathbf{II}_1$ algebra is universal, as in quantum field theory.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- Another fundamental fact — more of less equivalent to the universal divergence in the entanglement entropy — is that the type $\mathbf{II}_1$ algebra $\mathfrak{A}$ has **no** irreducible representation!

- By construction, $\mathfrak{A}$ acts on $\mathscr{H}$. But this is far from irreducible as it commutes with the action of $\mathfrak{B}$ on the same Hilbert space.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- One may make a smaller representation of $\mathfrak{A}$ by projecting $\mathscr{H}$ onto a invariant subspace by setting J_2 and considering the element of $\mathfrak{B}$ as following:

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

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$$\Pi_k' = \underbrace{J_2 \otimes J_2 \otimes \cdots \otimes J_2}_{first\ k\ terms} \otimes \mathbf{1}_{2\times 2} \otimes \mathbf{1}_{2\times 2} \cdots$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

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$$\mathbf{Tr}(\Pi'_k) = \langle \Psi | J_2 \otimes J_2 \otimes \cdots \otimes J_2 \otimes \mathbf{1}_{2\times2} \otimes \mathbf{1}_{2\times2} \cdots | \Psi \rangle = 2^{-k}$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- Because $\Pi_k'^2 = \Pi_k'$, it is a projection operator.

- The subspace $\mathscr{H}\Pi_k' \subset \mathscr{H}$ is a representation of $\mathfrak{A}$.

- In a sense that was made precisely by Murray and von Neumann, it is smaller than $\mathscr{H}$ by a factor of 2^k.

- Unfortunately, it is also (infinitely) far away from an irreducible representation.

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$$\begin{pmatrix} v_{11}^{[1]} & 0 \\ v_{21}^{[1]} & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} v_{11}^{[k]} & 0 \\ v_{21}^{[k]} & 0 \end{pmatrix} \otimes v^{[k+1]} \otimes \cdots \otimes v^{[n]} \otimes \cdots$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- The type $\mathbf{II}_1$ algebra we have considered have some properties in common with local algebras in quantum field theory.

- They both have a universal divergence in the entanglement entropy and do not have any irreducible representation.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

III. Algebras of type II

- The type $\mathbf{II}_1$ algebra we have considered have some properties in common with local algebras in quantum field theory.

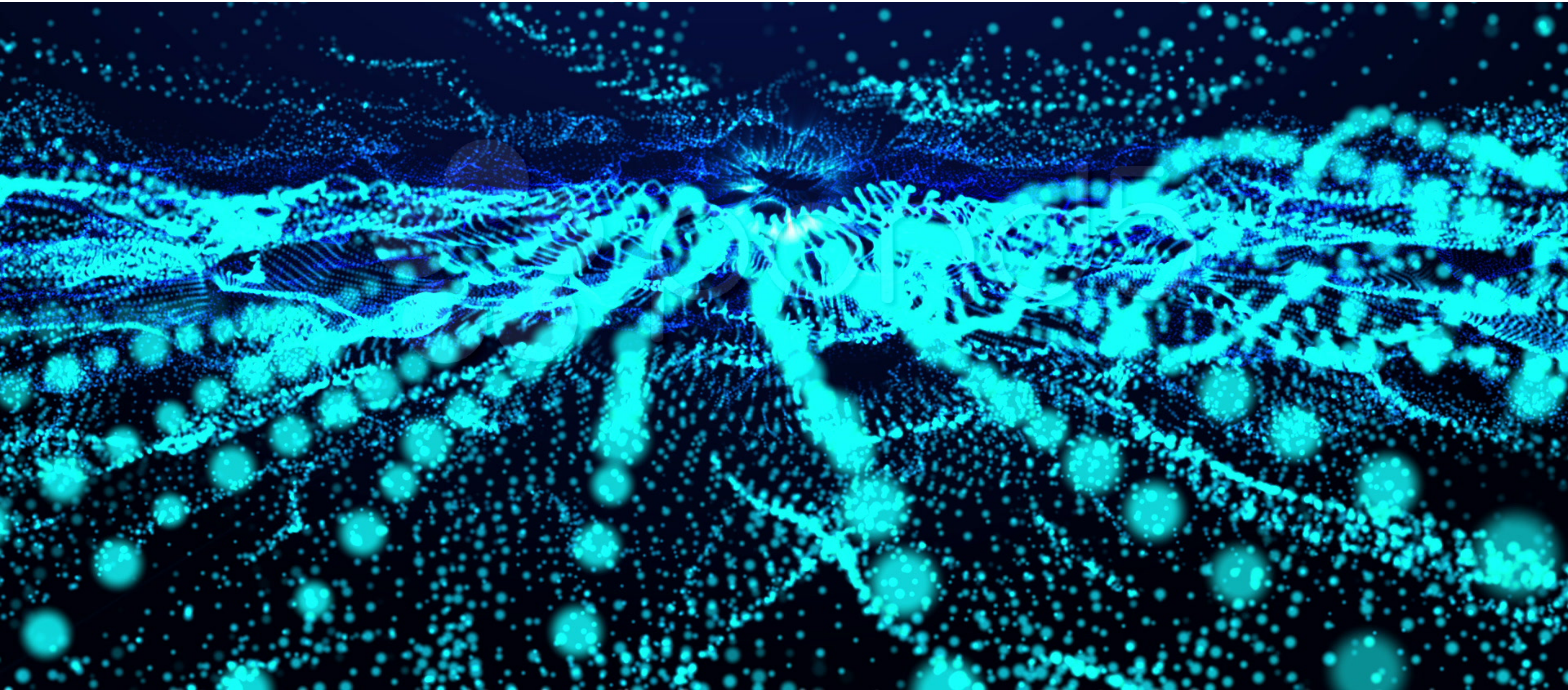
- They both have a universal divergence in the entanglement entropy and do not have any irreducible representation.

- But the local algebras in quantum field theory are **not** type $\mathbf{II}_1$ algebras, because they do **not** possess a trace!

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

IV. Algebras of type III

- More general algebras can be constructed by proceeding similarly, but with reduced entanglement.



ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

IV. Algebras of type III

- More general algebras can be constructed by proceeding similarly, but with reduced entanglement.

- One replaces the maximal entanglement limit element $\mathbf{1}'_{2\times 2}$ with $K_{2,\lambda}$, a pair of qubits with nonzero but nonmaximal entanglement.

$$K_{2,\lambda} = \frac{1}{\sqrt{1+\lambda}} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \quad 0 < \lambda < 1$$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

IV. Algebras of type III

- Then one can define the Hilbert space $\mathscr{H}_\lambda$ and the algebra $\mathfrak{A}_\lambda$ similarly to the type $\mathbf{II}_1$ case. $\mathfrak{A}_\lambda$ is different from $\mathfrak{A}$ because $\mathscr{H}_\lambda$ is different from $\mathscr{H}$.

- The definition of $\mathfrak{B}_\lambda$ is also similar to $\mathfrak{B}$, and $\Psi_\lambda = K_{2,\lambda} \otimes K_{2,\lambda} \otimes \cdots$ is again a cyclic and separating element for both $\mathfrak{A}_\lambda$ and $\mathfrak{B}_\lambda$.

- Unfortunately, the linear function $F(a_{\mathfrak{A}_\lambda}) = \langle \Psi_\lambda | a_{\mathfrak{A}_\lambda} | \Psi_\lambda \rangle$ does not satisfy $F(a_{\mathfrak{A}_\lambda} b_{\mathfrak{A}_\lambda}) = F(b_{\mathfrak{A}_\lambda} a_{\mathfrak{A}_\lambda})$, so it is not a trace.

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$$F(a_{\mathfrak{A}_\lambda} b_{\mathfrak{A}_\lambda}) = \langle \Psi_\lambda | a_{\mathfrak{A}_\lambda} b_{\mathfrak{A}_\lambda} | \Psi_\lambda \rangle = \prod_{i=1}^{k<\infty} \mathbf{Tr}(a^{[i]}_{\mathfrak{A}_\lambda} b^{[i]}_{\mathfrak{A}_\lambda} K^2_{2,\lambda}) \neq \prod_{i=1}^{k<\infty} \mathbf{Tr}(b^{[i]}_{\mathfrak{A}_\lambda} a^{[i]}_{\mathfrak{A}_\lambda} K^2_{2,\lambda})$$

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$$= \langle \Psi_\lambda | b_{\mathfrak{A}_\lambda} a_{\mathfrak{A}_\lambda} | \Psi_\lambda \rangle = F(b_{\mathfrak{A}_\lambda} a_{\mathfrak{A}_\lambda})$$

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- **Indeed the algebra $\mathfrak{A}_\lambda$ does not admit a trace.**

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

IV. Algebras of type III

- The entanglement entropy in the state Ψ_λ is divergent, because it describes an infinite collection of qubit pairs each with the same entanglement.

- Any state in $\mathscr{H}_\lambda$ has the same universal leading divergence in the entanglement entropy.

- The action of $\mathfrak{A}_\lambda$ on $\mathscr{H}_\lambda$ is again far away from irreducible.

- However, although we will not prove it, the invariant subspaces in which $\mathscr{H}_\lambda$ can be decomposed are isomorphic as representations of $\mathfrak{A}_\lambda$ to $\mathscr{H}_\lambda$ itself: **a hyperfinite von Neumann algebra of type $\mathbf{III}$ has only one nontrivial representation, up to isomorphism**.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

IV. Algebras of type III

- For $\lambda \neq \tilde{\lambda}$, $\mathfrak{A}_\lambda$ and $\mathfrak{A}_{\tilde{\lambda}}$ are nonisomorphic.
- Other cases?

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

IV. Algebras of type III

- We have seen the infinite entanglement chain with fixed $\lambda = 1$ (maximal entanglement, type $\mathbf{II}_1$) and fixed $0 < \lambda < 1$ (nonmaximal entanglement, type $\mathbf{III}$).

- Given a sequence $\{\lambda_n\}$, $0 < \lambda_n \leqslant 1$, and consider the algebra $\mathfrak{A}_{\vec{\lambda}}$ acts on the left of the Hilbert space $\mathscr{H}_{\vec{\lambda}}$ completed from the vectors $v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes \cdots$ such that $v_n = K_{2,\lambda_n}$ for all but finitely many n.

- The vector $\Psi_{\vec{\lambda}} = K_{2,\lambda_1} \otimes K_{2,\lambda_2} \otimes \cdots \otimes K_{2,\lambda_n} \otimes \cdots$ is again a cyclic and separating vector for $\mathfrak{A}_{\vec{\lambda}}$ and $\mathfrak{A}'_{\vec{\lambda}} = \mathfrak{B}_{\vec{\lambda}}$.

- The expectation $\langle \Psi_{\vec{\lambda}} | a | \Psi_{\vec{\lambda}} \rangle$ is not a trace unless the λ_i are all 1.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

IV. Algebras of type III

- There are some cases as following:

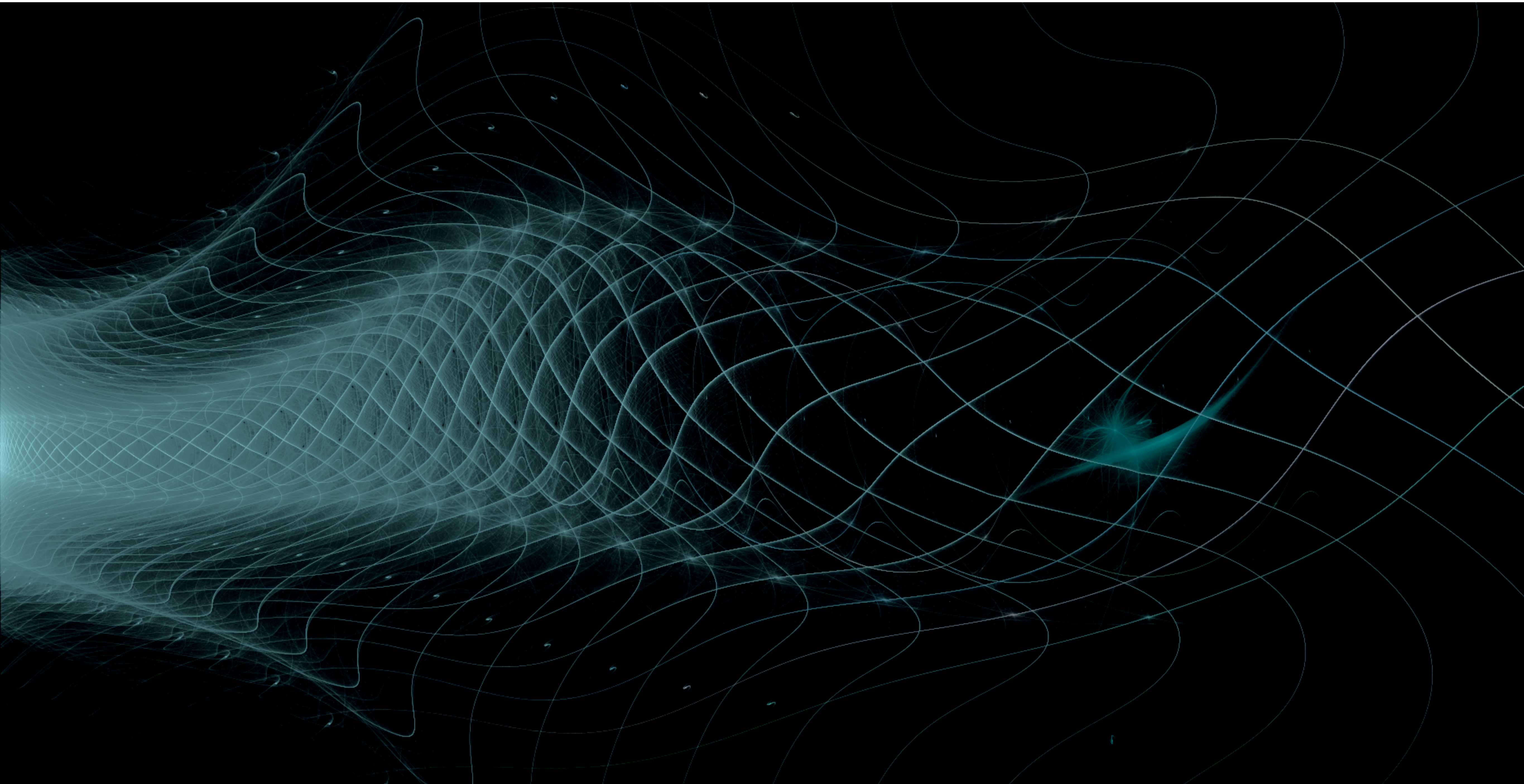
  1. $\lim \lambda_n \to \lambda,\ 0 < \lambda < 1$: this gives the type $\mathbf{III}_\lambda$ algebra as before;

  2. $\lim \lambda_n \to 0$: if the convergence is fast enough, this gives the type $\mathbf{I}_\infty$ algebra; if the convergence is not fast enough, this gives a new algebra defined to be of type $\mathbf{III}_0$;

  3. $\{\lambda_n\}$ does not converge and has at least two limit points in $(0,\ 1)$: this is a new algebra defined to be of type $\mathbf{III}_1$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory



ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- Conclusion: the local algebras $\mathfrak{A}(\mathscr{U})$ in quantum field theory are of type $\mathbf{III}$, because they do not have a trace.

- They are believed to be of type $\mathbf{III}_1$.

- The aim of this subsection is to give a somewhat heuristic explanation of this statement by using the **spectrum of the modular operator** to distinguish the different algebras.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- The basic block of the infinite type $\mathbf{III}$ algebras are the bipartite system V with a pair algebras M_A and M_B acting on the left and right on V.

- Let us consider the cyclic and separating vector $K_{2,\lambda}$ for M_A and M_B,

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$$\rho_A = \mathbf{Tr}_B | K_{2,\lambda} \rangle \langle K_{2,\lambda} | = K_{2,\lambda} K^{\dagger}_{2,\lambda} = \frac{1}{1+\lambda} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

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$$\rho_B = \mathbf{Tr}_A |K_{2,\lambda}\rangle\langle K_{2,\lambda}| = K_{2,\lambda}^{\dagger} K_{2,\lambda} = \frac{1}{1+\lambda} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

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- In fact, Δ_Ψ is diagonalized.

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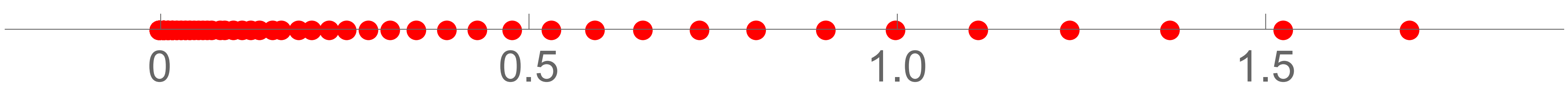
- The eigenvalues of this linear transformation is $1,\ \lambda,\ \lambda^{-1}$.
- In fact, Δ_Ψ is diagonalized.
- The whole Hilbert space $\mathscr{H}_\lambda$ is an infinite tensor product of the bipartite system.
- So the eigenvalues of Δ_Ψ are all integer powers of λ, each occurring infinitely often.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- The accumulation points of the eigenvalues are $\{0\} \cup \{\lambda^n | n \in \mathbb{Z}\}$.



ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- $\Psi_{\vec{\lambda}} = K_{2,\lambda_1} \otimes K_{2,\lambda_2} \otimes \cdots \otimes K_{2,\lambda_n} \otimes \cdots$ is cyclic separating for $\mathfrak{A}_\lambda$ if the $\{\lambda_n\}$ approach λ sufficiently fast.

- The spectrum of $\Delta_{\Psi_{\vec{\lambda}}}$ is more complicated, but 0 and the integer powers of λ are still accumulation points.

- Still more generally, in the case of a type $\mathbf{III}_\lambda$ algebra, for any cyclic separating vector Ψ, not necessarily of the form $\Psi_{\vec{\lambda}}$, the integer powers of λ and 0 are accumulation points of the eigenvalues.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- For type $\mathbf{III}_0$ algebra, $\lim \lambda_n \to 0$, so the only unavoidable accumulation points of the eigenvalues of $\Delta_{\Psi_{\vec{\lambda}}}$ are 0 and 1.

- These values continue to be accumulation points if $\Psi_{\vec{\lambda}}$ is replaced by any cyclic separating vector of a type $\mathbf{III}_0$ algebra.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- Type $\mathbf{III}_1$ algebra ($\{\lambda_n\}$ has more than one limit points):
  - Suppose λ_n take two values λ and $\tilde{\lambda}$, each for infinite times;
  - The eigenvalues of $\Delta_{\Psi_{\vec{\lambda}}}$ consist of the numbers $\lambda^n \tilde{\lambda}^m$ ($m, n \in \mathbb{Z}$), each value occurring infinitely many times;
  - If there is a $0 < \lambda' < 1$ s.t. $\lambda = \lambda'^a$, $\tilde{\lambda} = \lambda'^b$ ($a, b \in \mathbb{Z}$) , then the algebra is in fact a type $\mathbf{III}_{\lambda'}$ algebra;
  - Otherwise, any non-negative real number is an accumulation point of the eigenvalues of the operator $\Delta_{\Psi_{\vec{\lambda}}}$.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- Type $\mathbf{III}_1$ algebra ($\{\lambda_n\}$ has more than one limit points):
  - For any cyclic separating vector Ψ, the spectrum of Δ_Ψ (including accumulation points of eigenvalues) comprises the full semi-infinite interval $[0, +\infty)$.

  - ($0.9^n \times 0.8^m,\ n, m \in \mathbb{Z}, -100 \leqslant n, m \leqslant 100$)

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- Quantum field theory:
  - Consider the wedge region $\mathscr{U}$ and the cyclic separating vector Ω, the modular operator is $\Delta_\Omega = \exp(-2\pi K)$.
  - The boost generator K has a continuous spectrum consisting of all real numbers, so Δ_Ω has a continuous spectrum $[0, +\infty)$ consisting of all positive numbers.
  - At short distances, any state is indistinguishable from the vacuum.
  - So we would expect that acting on excitations of very short wavelength, Δ_Ψ can be approximated by Δ_Ω and therefore has all points in $[0, +\infty)$ in its spectrum.

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- Quantum field theory:
  - The algebra $\mathfrak{A}(\mathscr{U})$ is of type $\mathbf{III}_1$.

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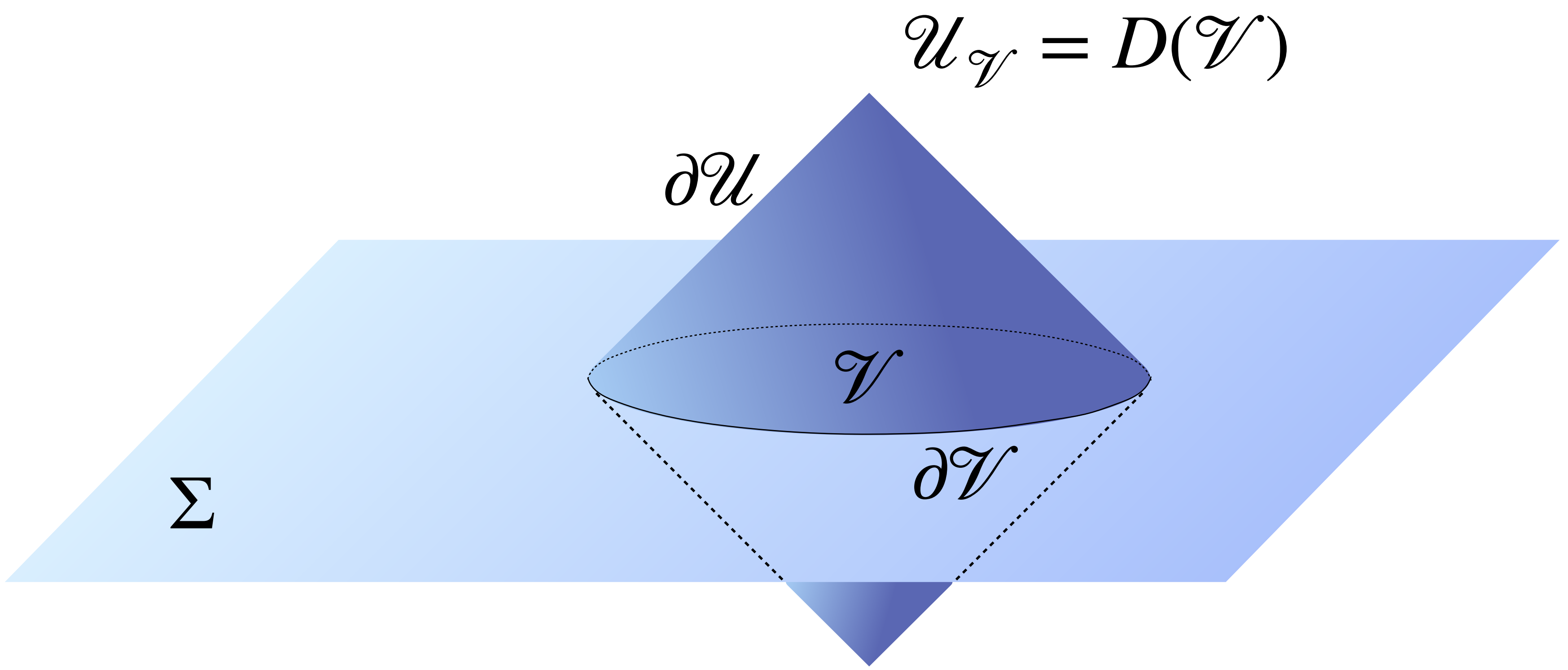

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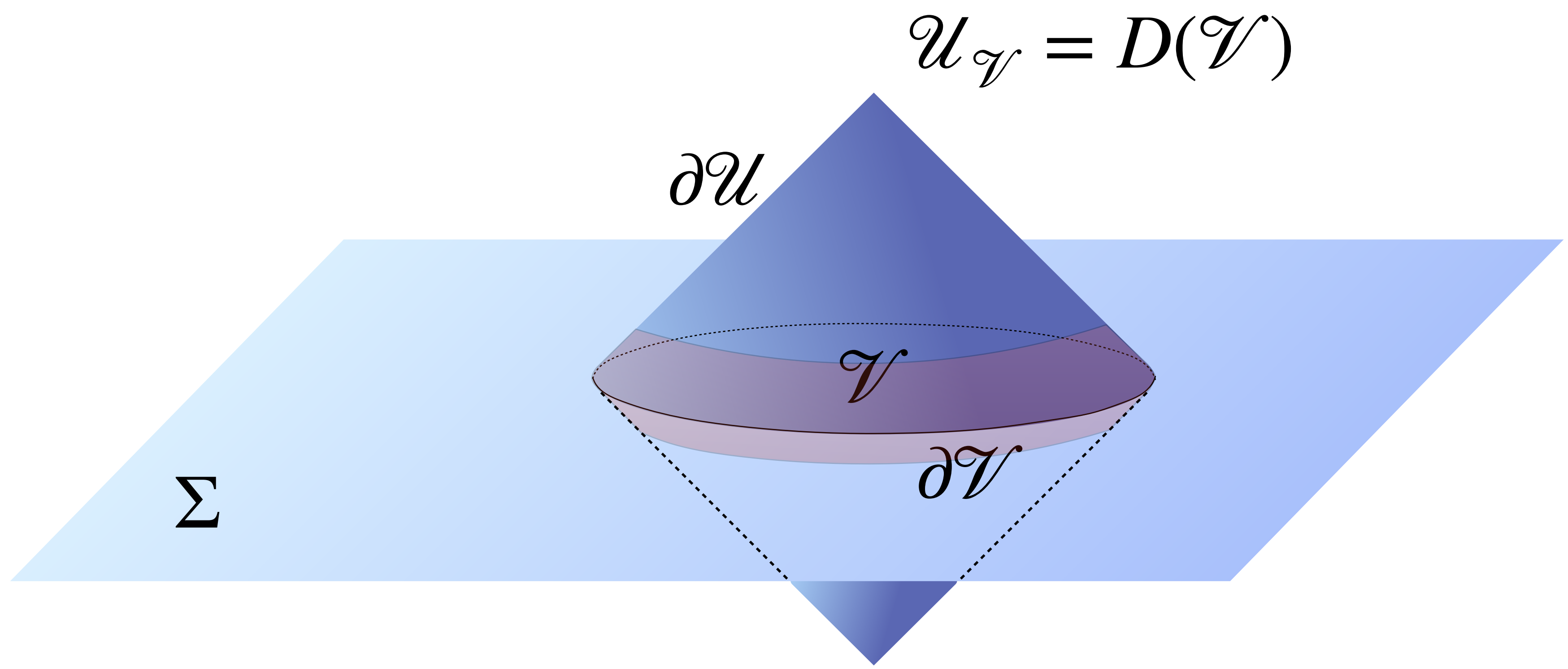

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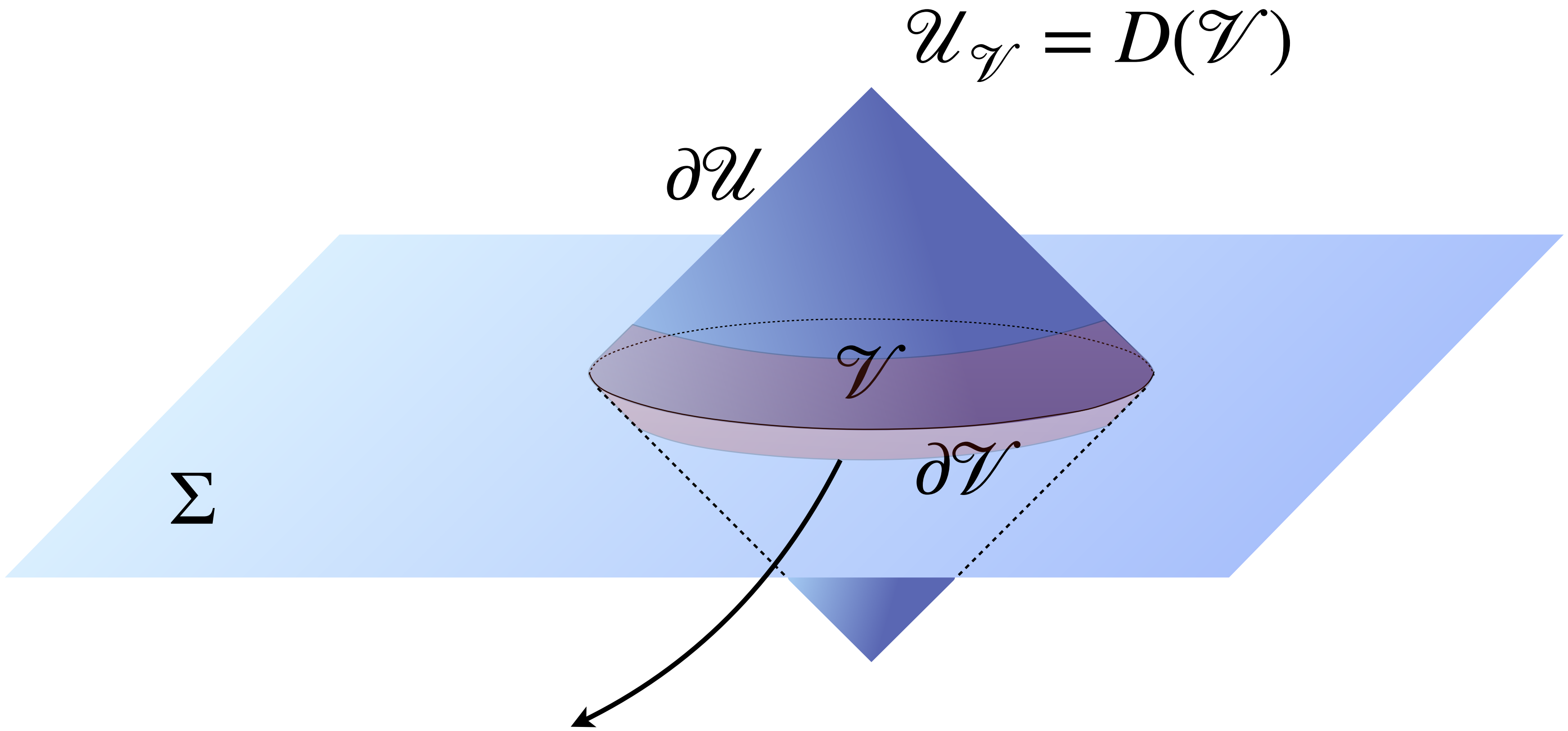
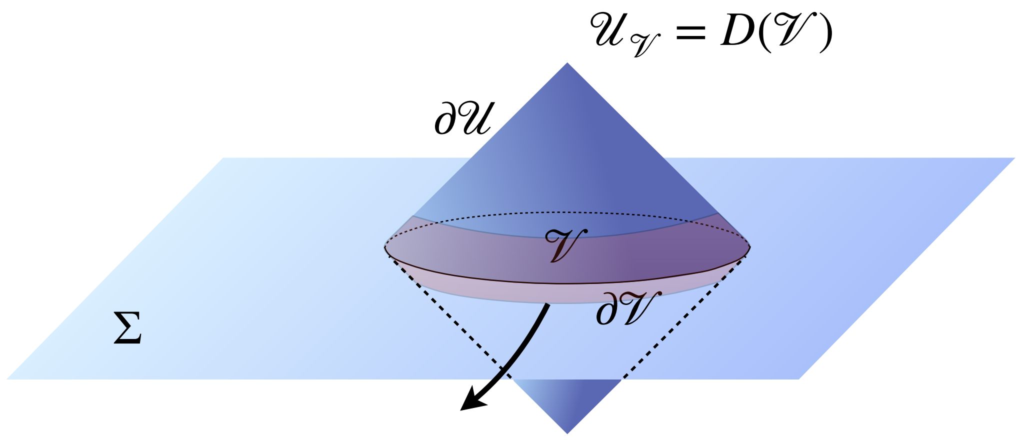


The corner of $\partial\mathscr{U}_{\mathscr{V}}$ along $\partial\mathscr{V}$

ALGEBRAS WITH A UNIVERSAL DIVERGENCE IN THE ENTANGLEMENT ENTROPY

V. Back to quantum field theory

- Quantum field theory:
    - Consider very high energy excitations localized near the corner
    - For these modes, $\mathscr{U}_{\mathscr{V}}$ looks like the wedge region $\mathscr{U}$
    - So one would expect that for such high energy excitations, $\Delta_{\Omega}(\mathscr{U}_{\mathscr{V}})$ looks like the Lorentz boost generators and has all positive real numbers in its spectrum



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