Entanglement properties of quantum field theory

A note of Witten's paper "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory"

Part I: The Reeh-Schlieder Theorem and Relative Entropy in Quantum Field Theory

Hao Zhang
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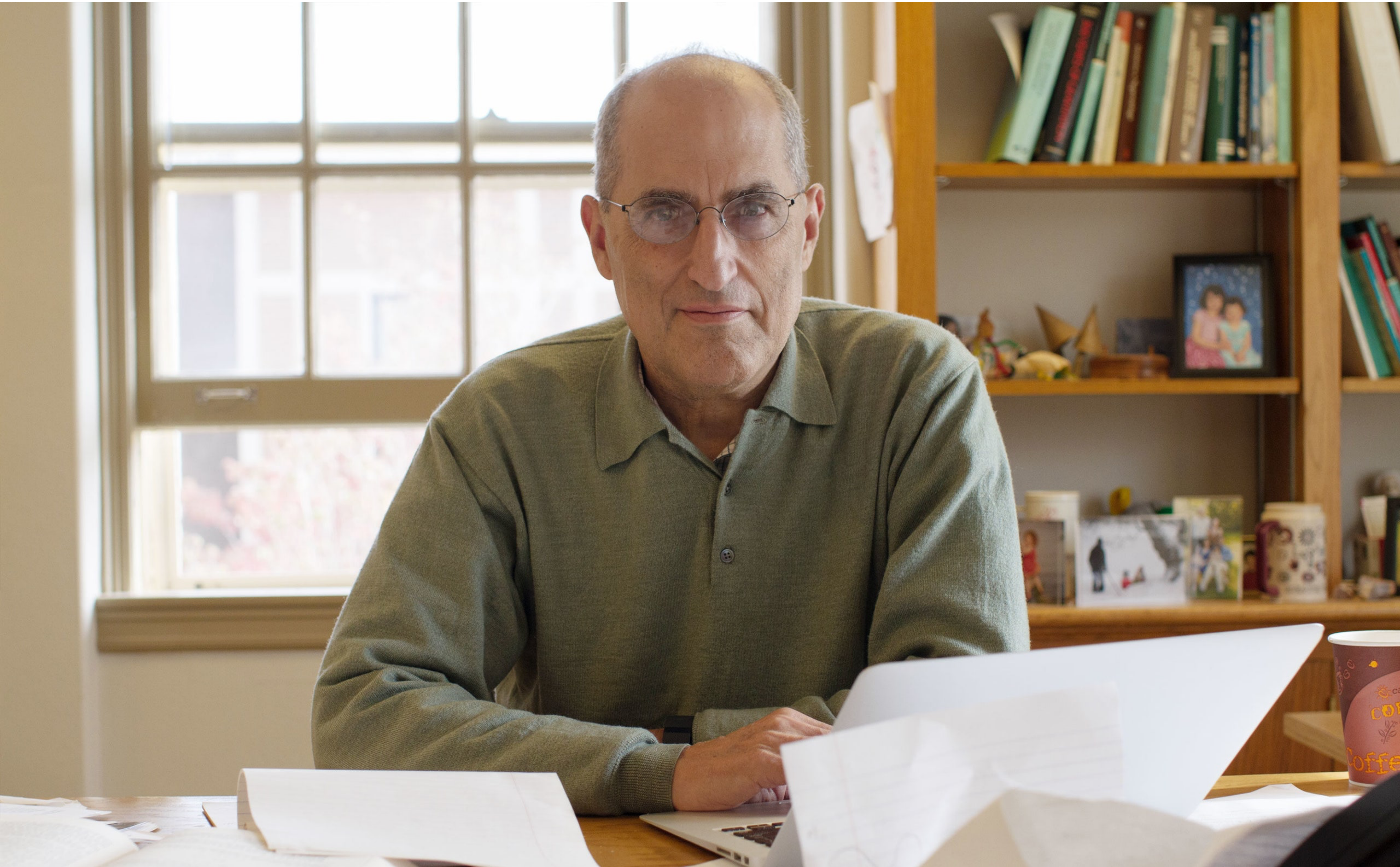
本文的目的与特点



本文的目的与特点

- 总结近年来量子信息与纠缠在量子场论和弦理论中扮演的日益重要的角色。
- 不是全面的综述，关注背后的（未被言明的）数学。
- 涉及一些公理化与代数量子场论的古典文献。

CONTENTS

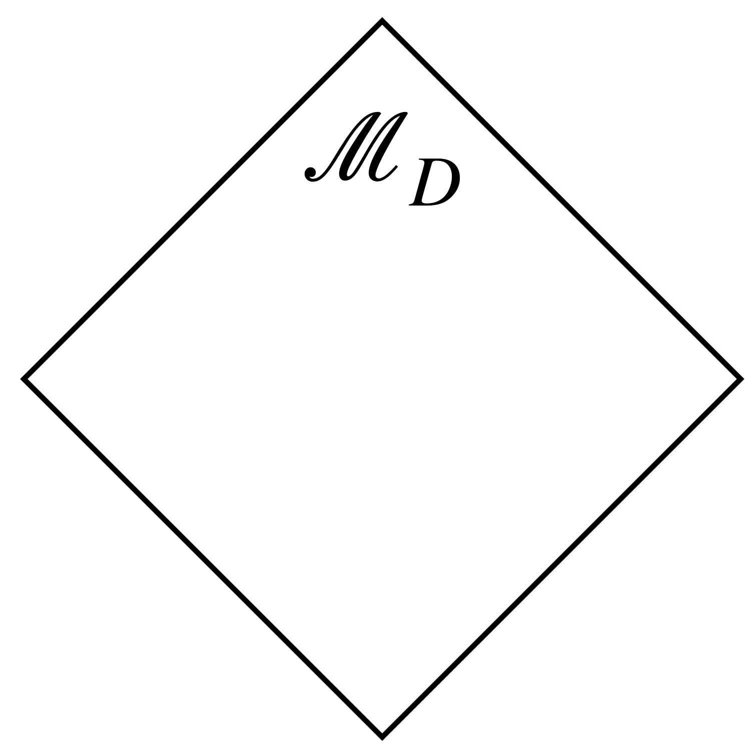
THE REEH-SCHLIEDER THEOREM

THE REEH-SCHLIEDER THEOREM

I. Statement

- Starting point: Consider a QFT in D-dimensional Minkowski spacetime $\mathscr{M}_D$, let $\mathscr{H}_0$ is the vacuum sector of the vacuum state Ω in the whole Hilbert space $\mathscr{H}$.

- $\phi(x)$ is quantum field (need not to be elementary), then the states $|\Psi_{\vec{f}}\rangle \equiv \phi_{f_1}\phi_{f_2}\cdots\phi_{f_n}|\Omega\rangle$ ($\phi_f \equiv \int d^D x f(x)\phi(x)$) are dense in $\mathscr{H}_0$.



THE REEH-SCHLIEDER THEOREM

I. Statement

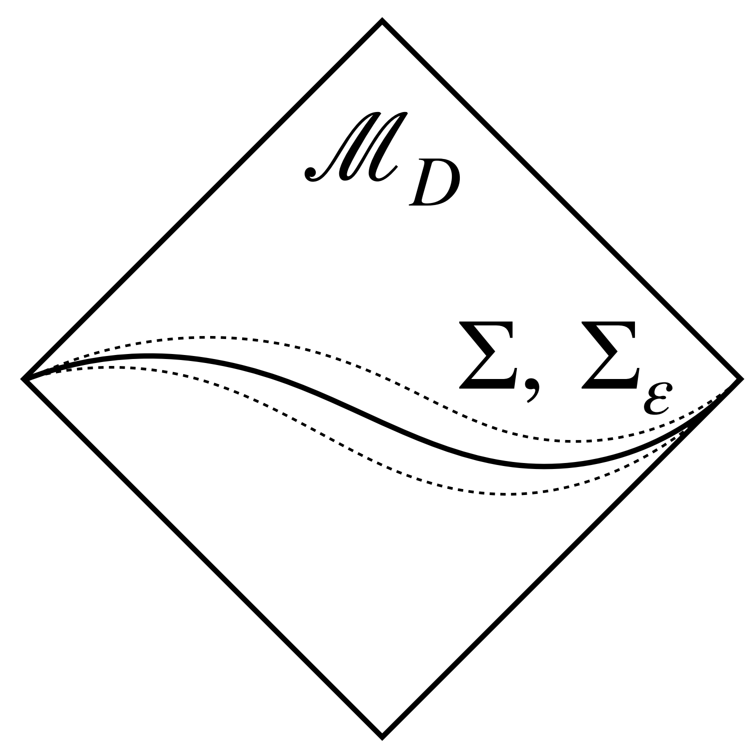
- Choosing smooth functions f_i supported in a small neighborhood of a Cauchy surface Σ,

$$\mathbf{supp}\, f_i \equiv \{x \in \mathscr{M}_D \,|\, f_i(x) \neq 0\} \subset \Sigma_\varepsilon \equiv \left\{ x \in \mathscr{M}_D \,\middle|\, \min_{z \in \Sigma} |x^0 - z^0| < \varepsilon \right\}$$

- The states $|\Psi_{\vec{f}}\rangle \equiv \phi_{f_1}\phi_{f_2}\cdots\phi_{f_n}|\Omega\rangle$ also span a dense subspace of $\mathscr{H}_0$.

- This is a quantum theory version of the completeness of the initial value problem (Cauchy problem).

- We won't prove it but a stronger result.

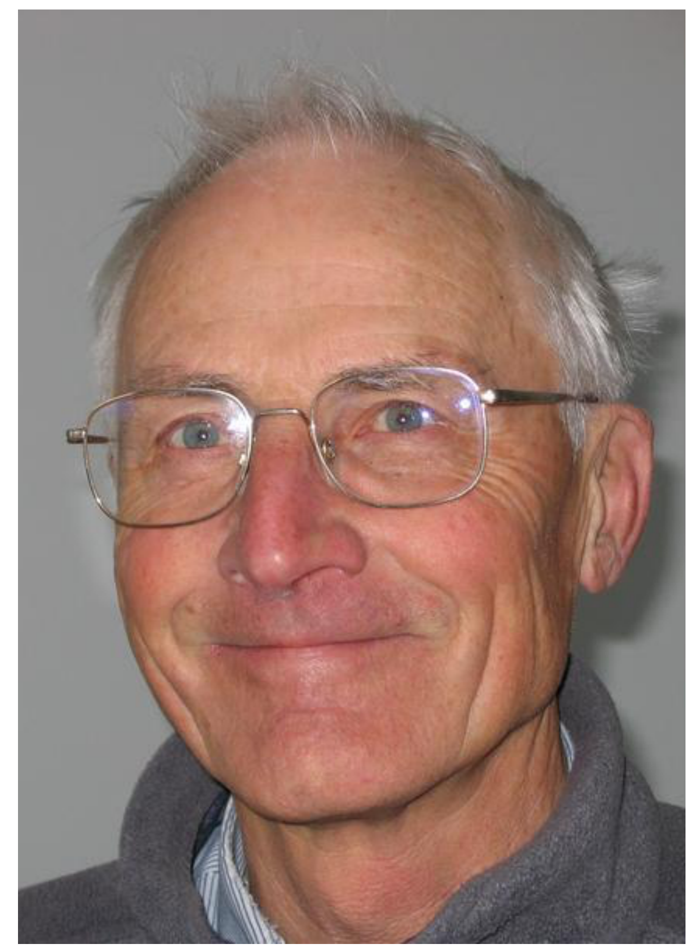


THE REEH-SCHLIEDER THEOREM

I. Statement

- The Reeh-Schlieder theorem (1961, *“Bemerkungen zur unitäräquivalenz von lorentzinvarianten feldern”*, Nuovo Cimento **22**, 1051-1068):

Siegfried Schlieder
(1918-2003)



Helmut Reeh

THE REEH-SCHLIEDER THEOREM

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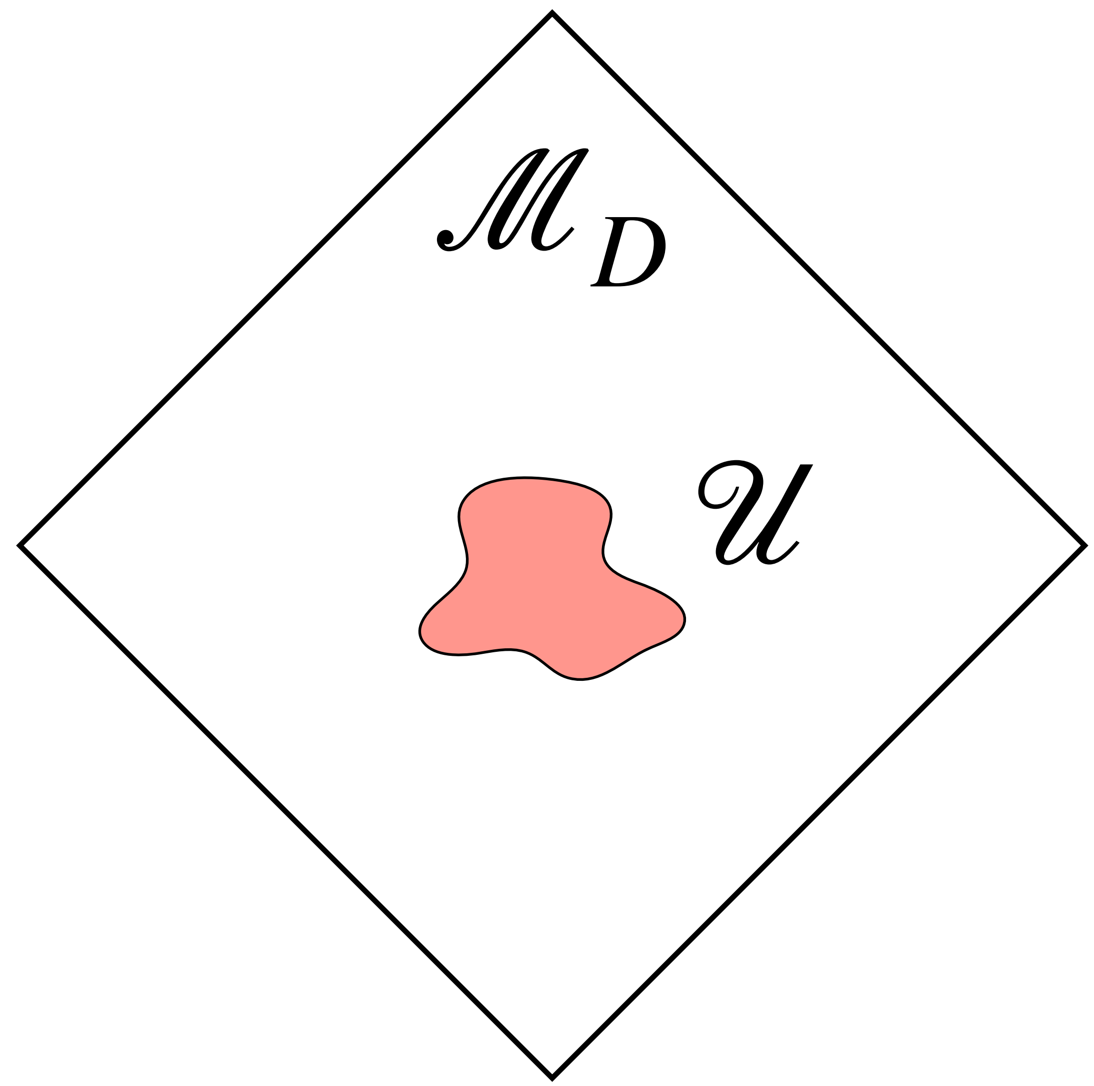
- The Reeh-Schlieder theorem (1961): for any open subset $\mathscr{U} \subset \mathscr{M}_D$, the subspace spanned by $|\Psi_{\vec{f}}\rangle \equiv \phi_{f_1}\phi_{f_2}\cdots\phi_{f_n}|\Omega\rangle$, where $\mathbf{supp}\, f_i \subset \mathscr{U}$, is dense in $\mathscr{H}_0$.



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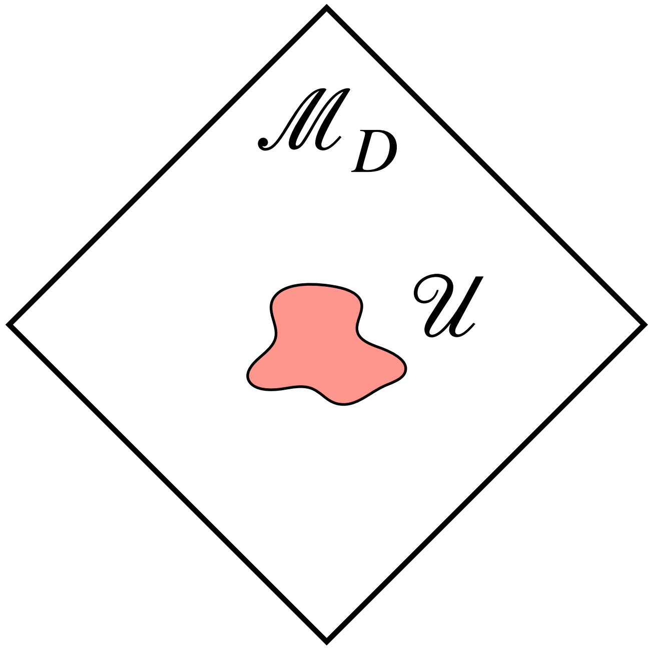

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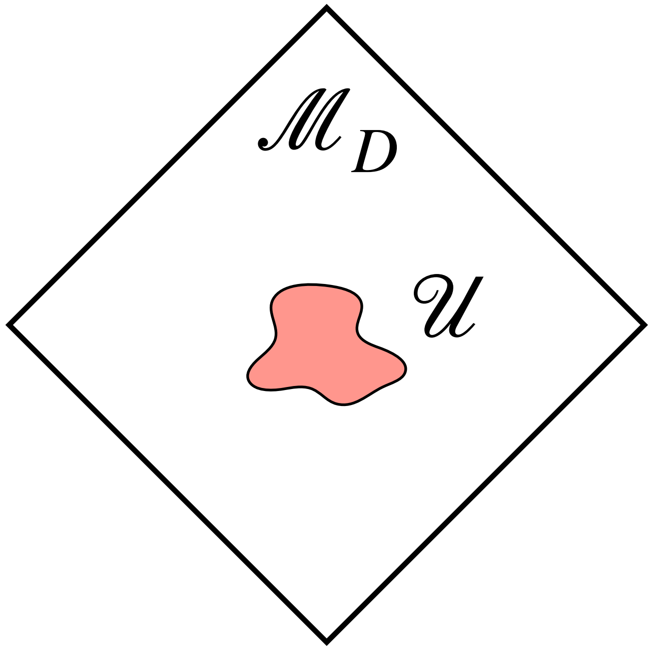

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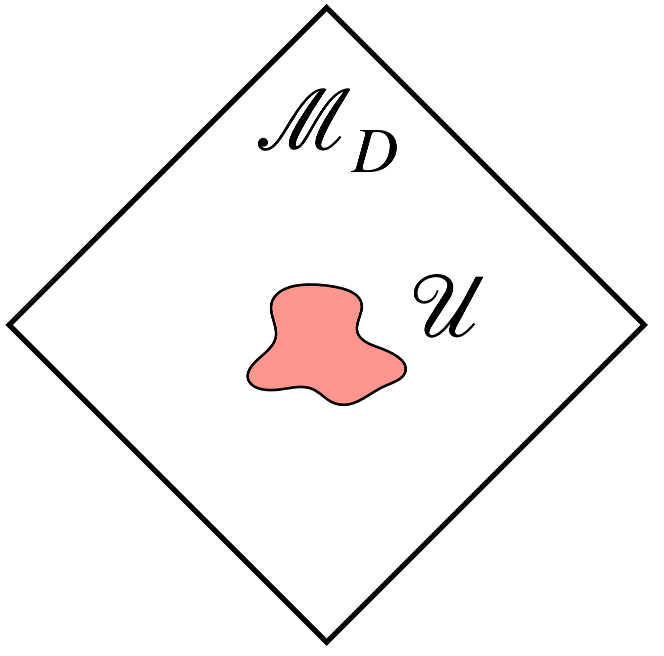

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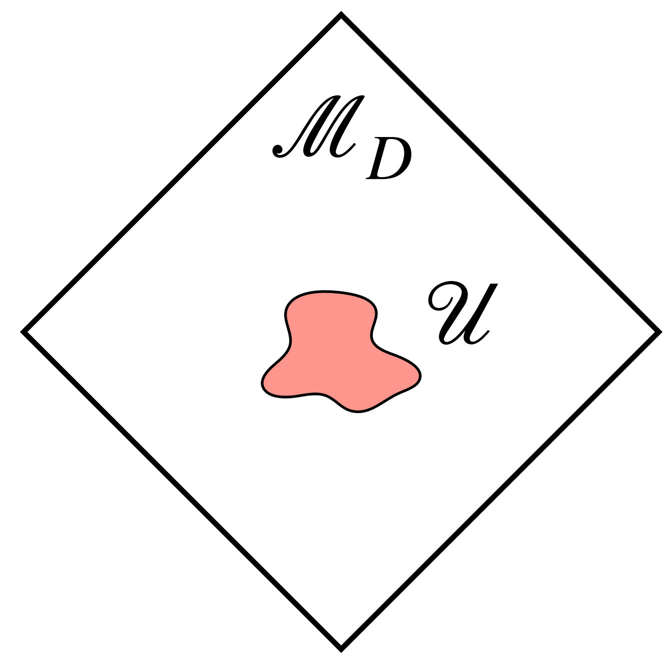

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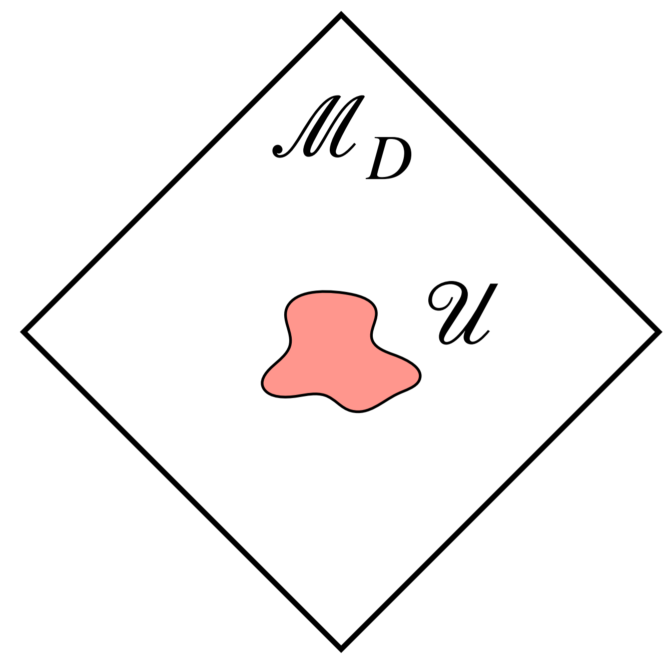

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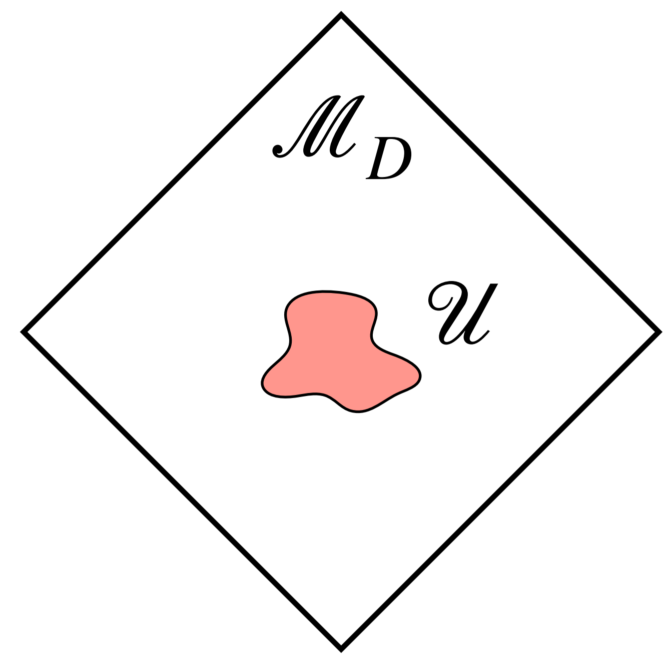

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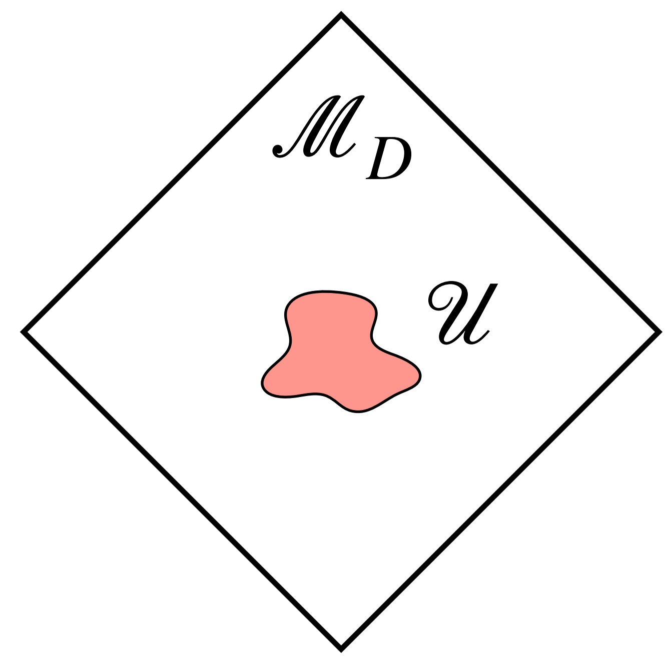

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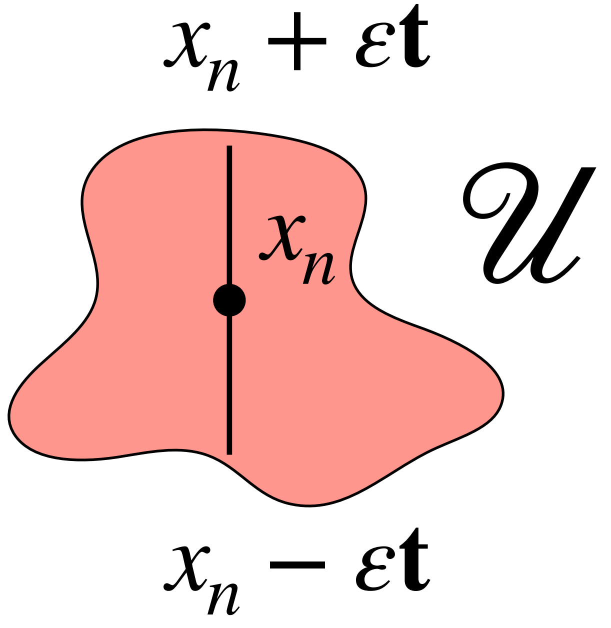
- In the upper half plane, we have



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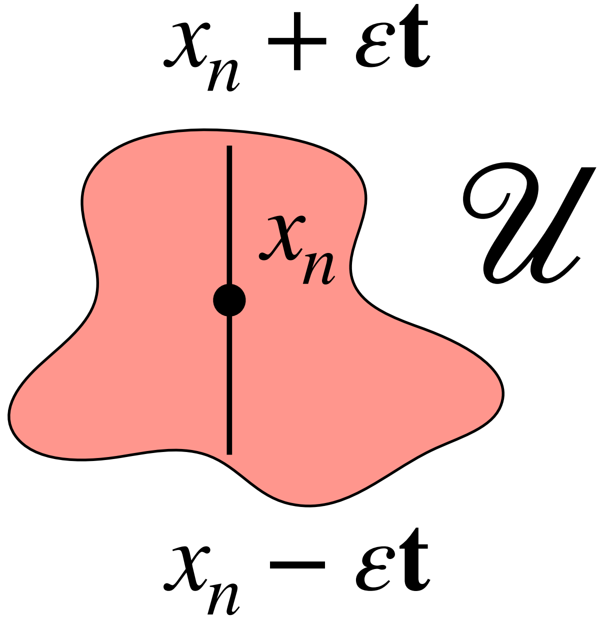

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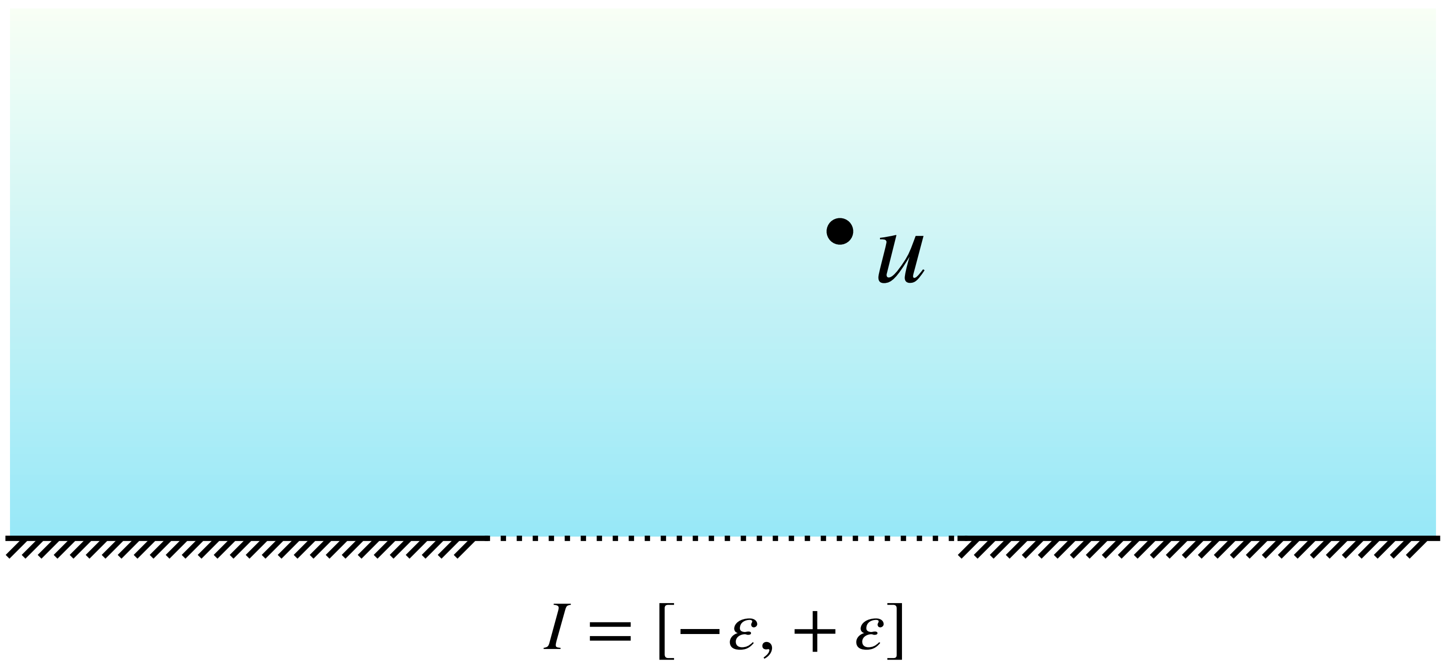

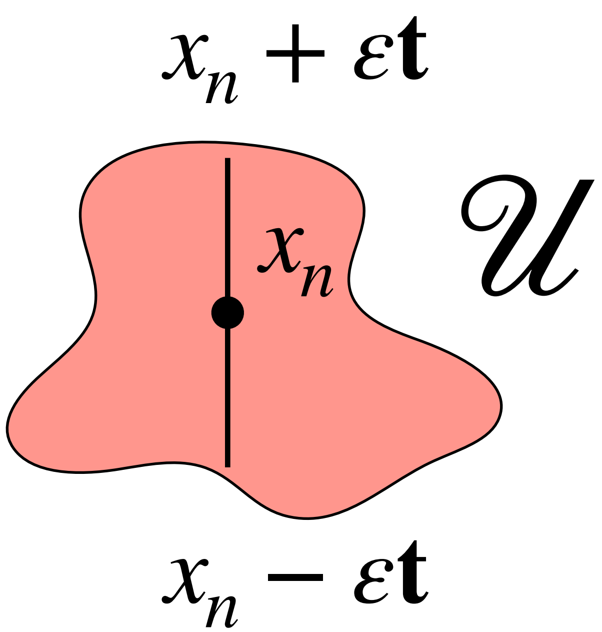

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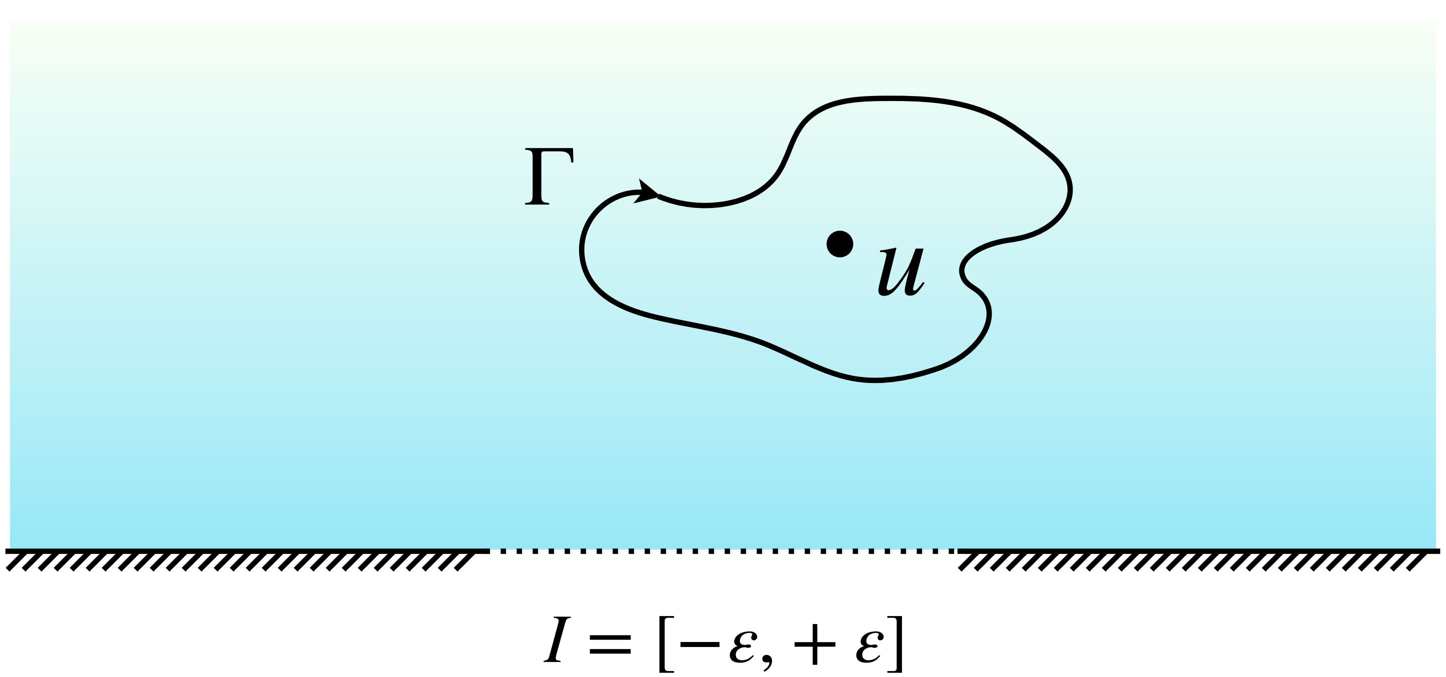

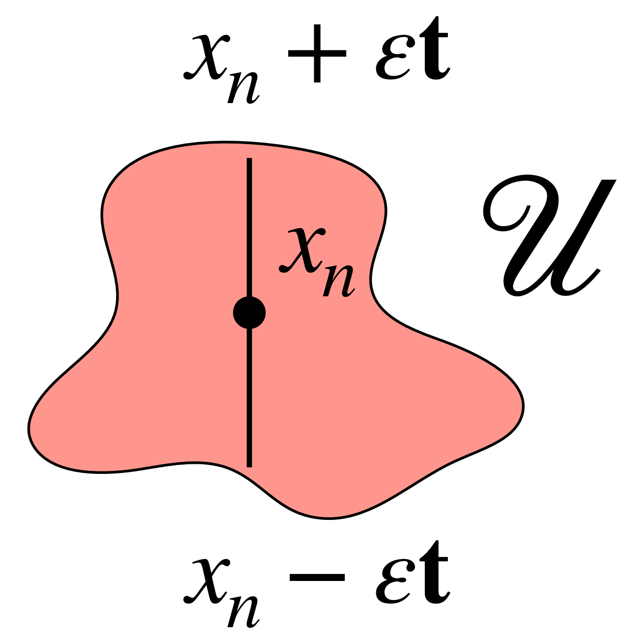

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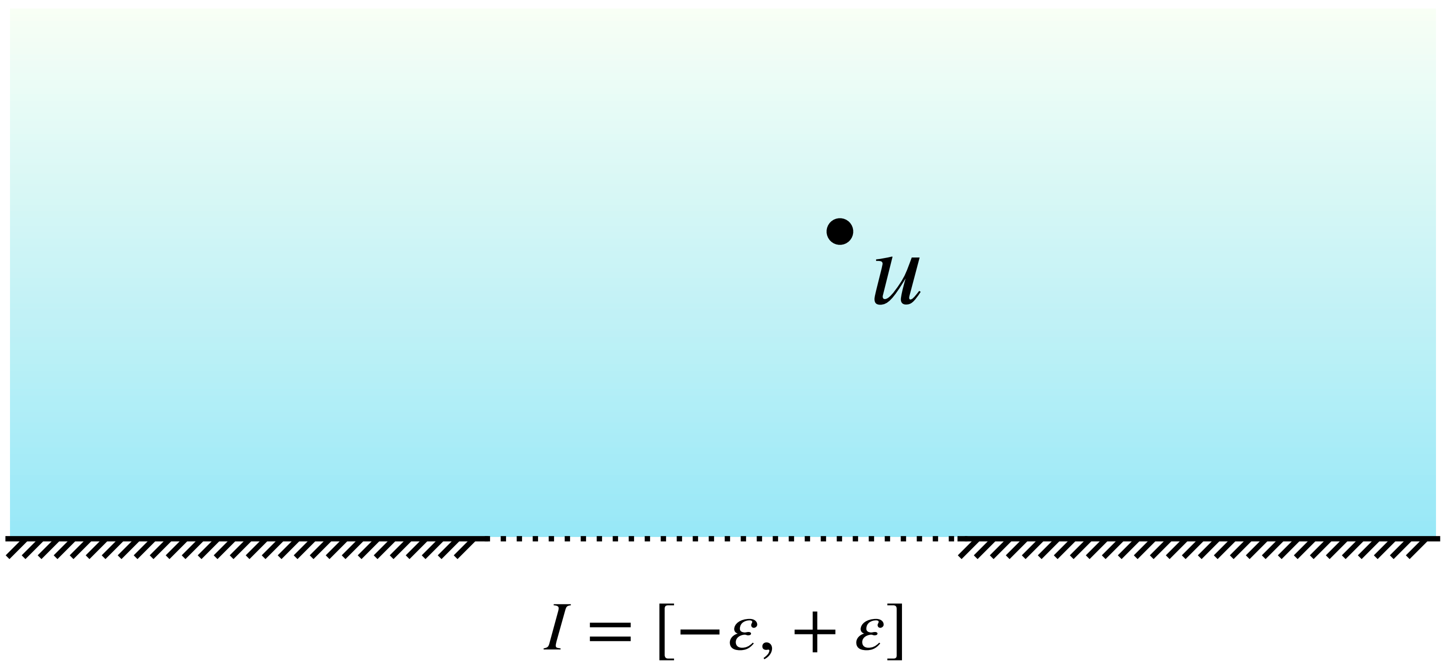

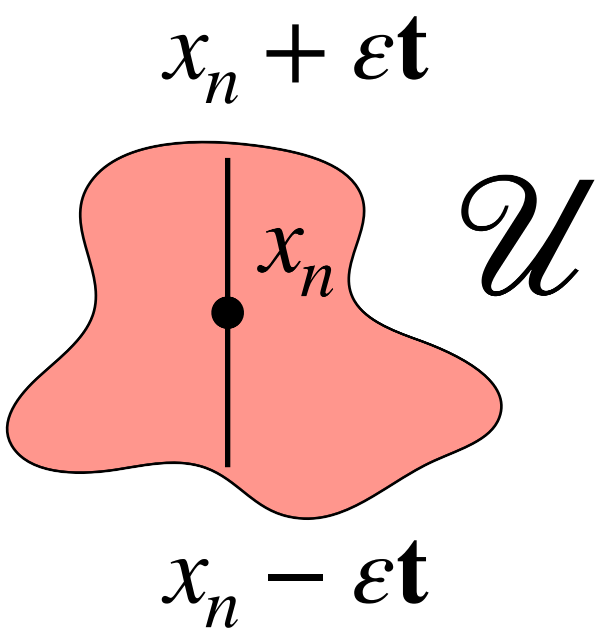

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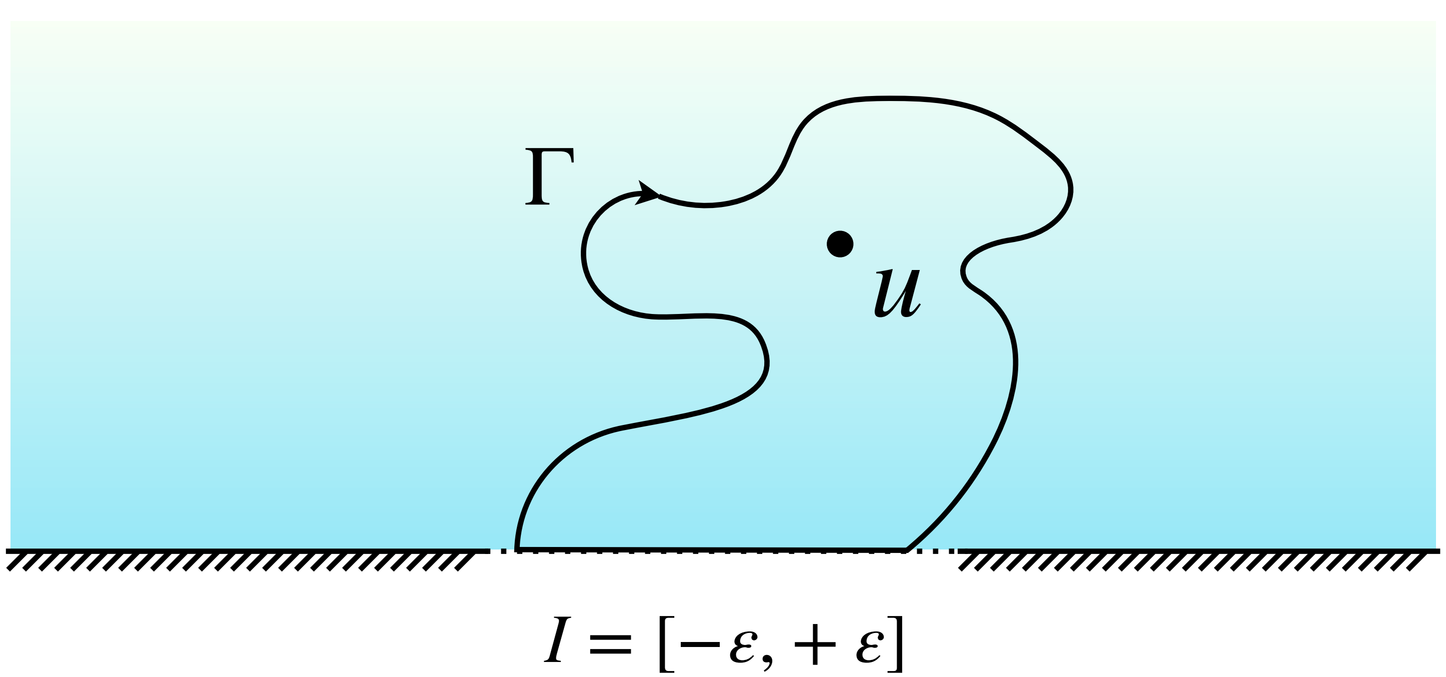

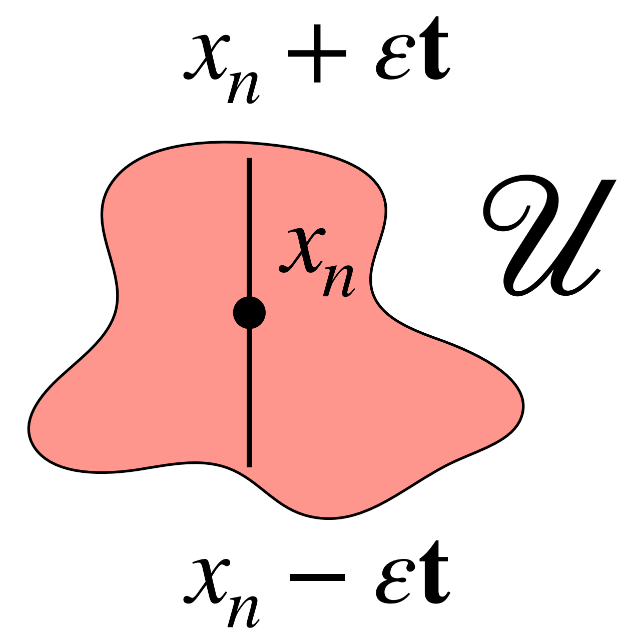

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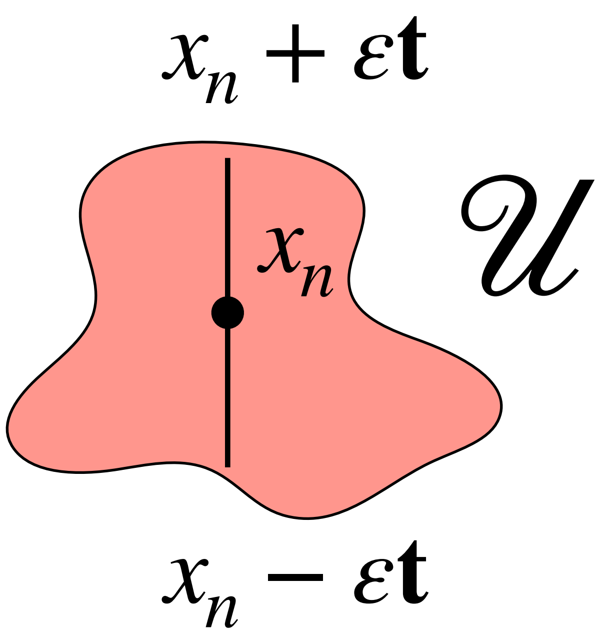

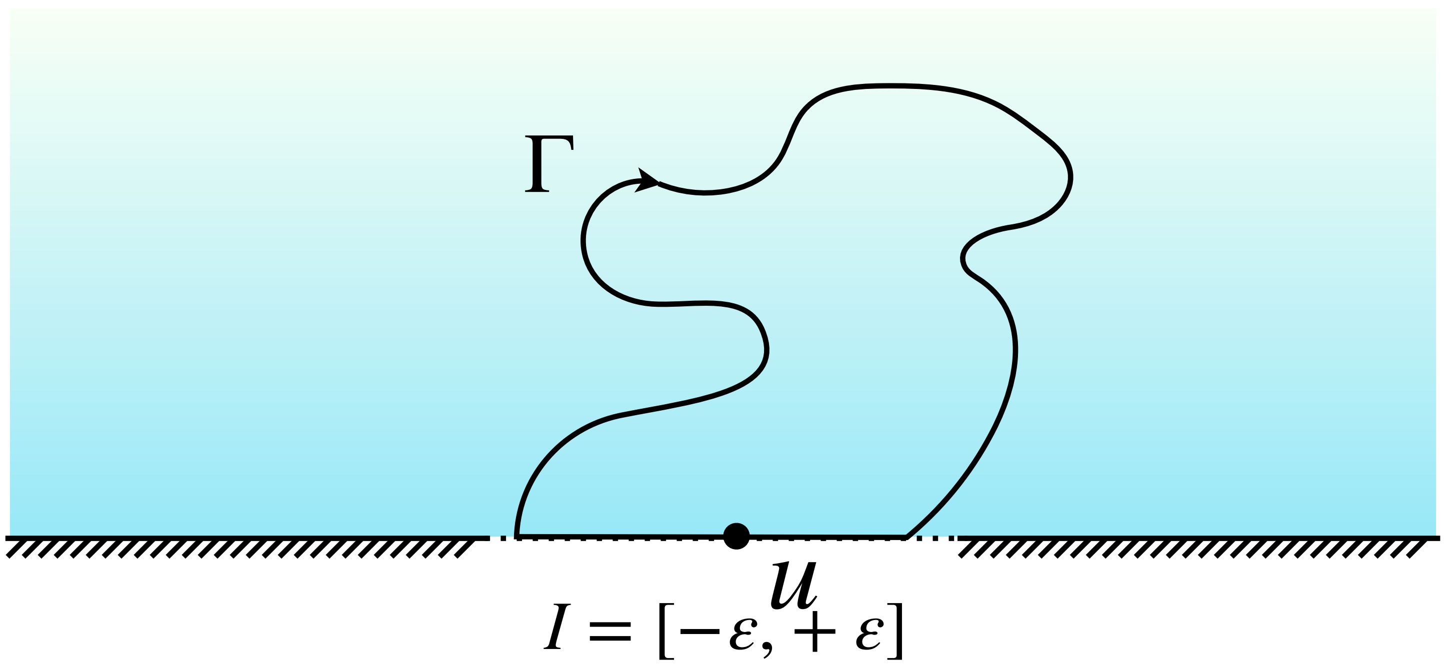

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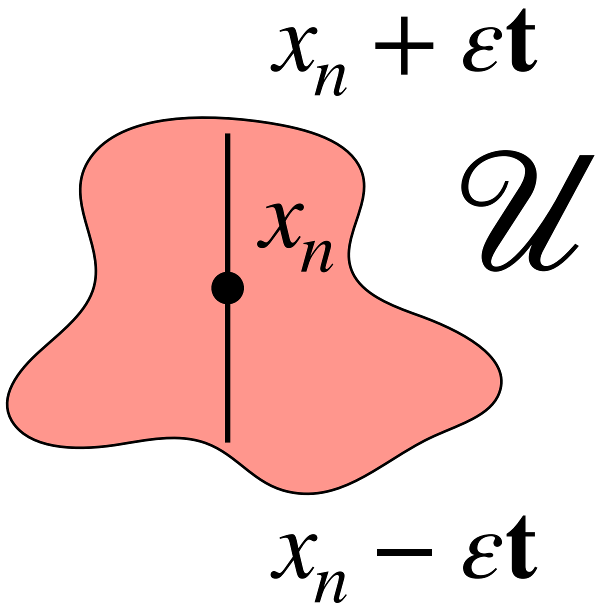

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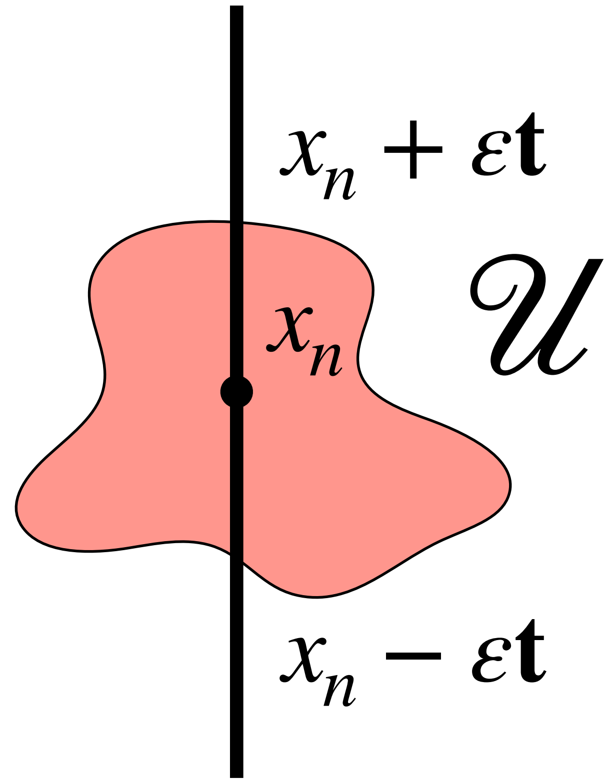

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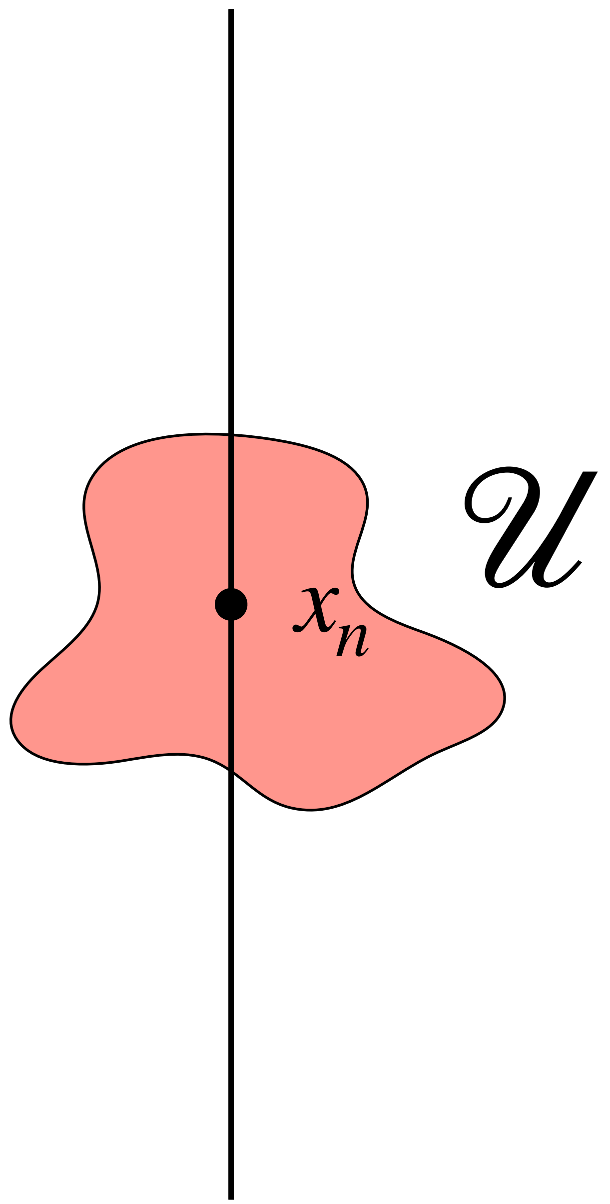

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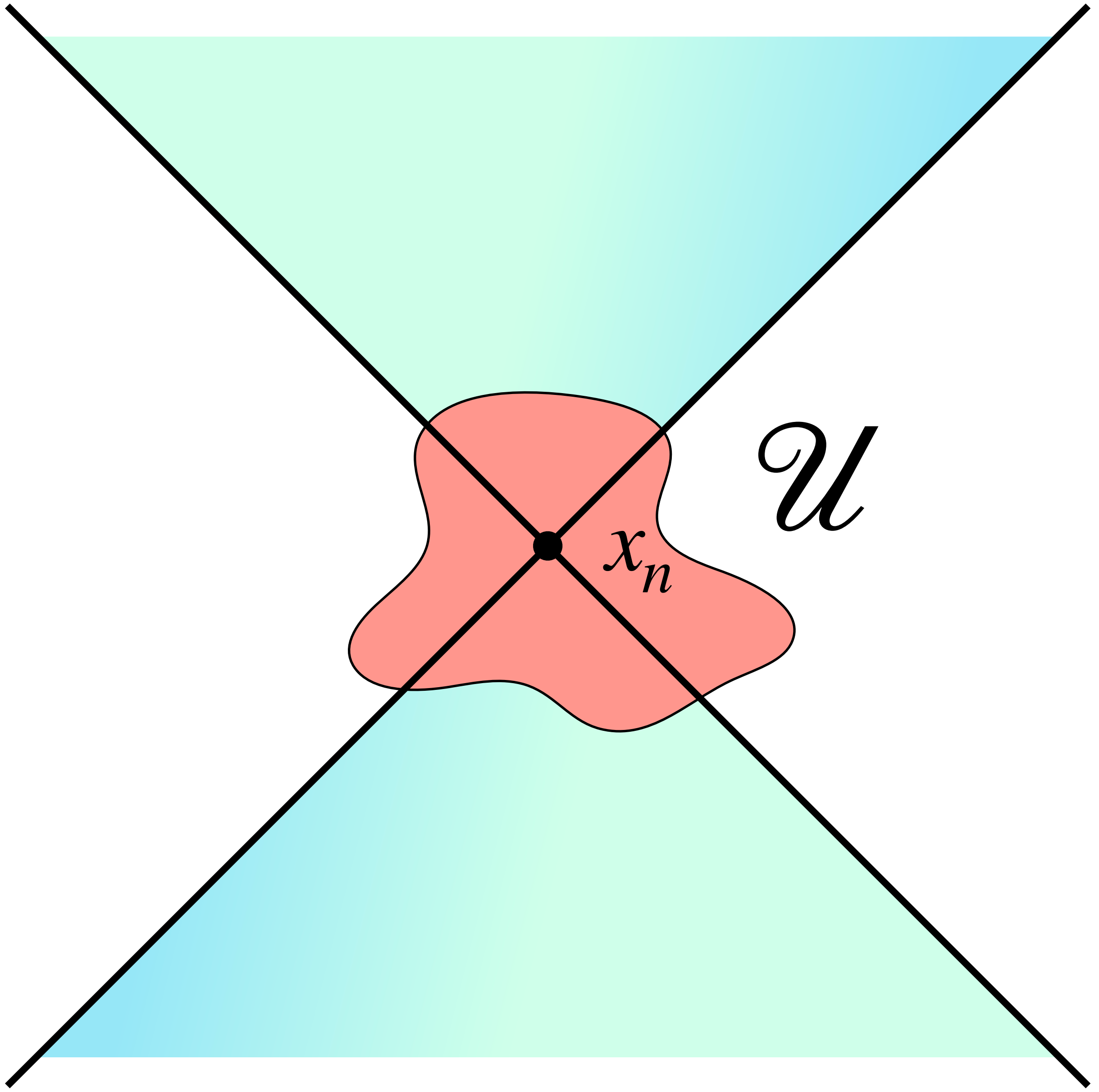

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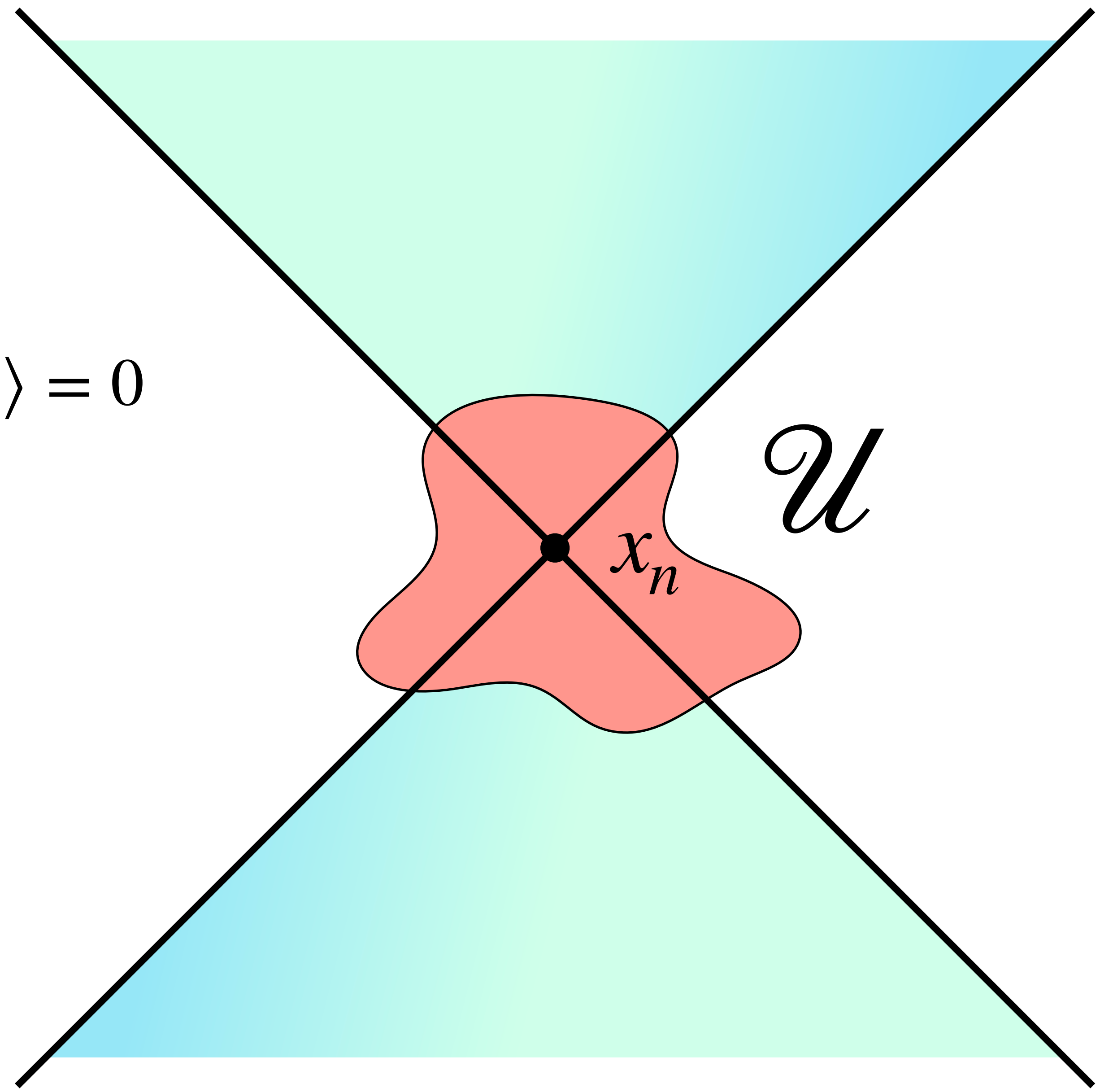

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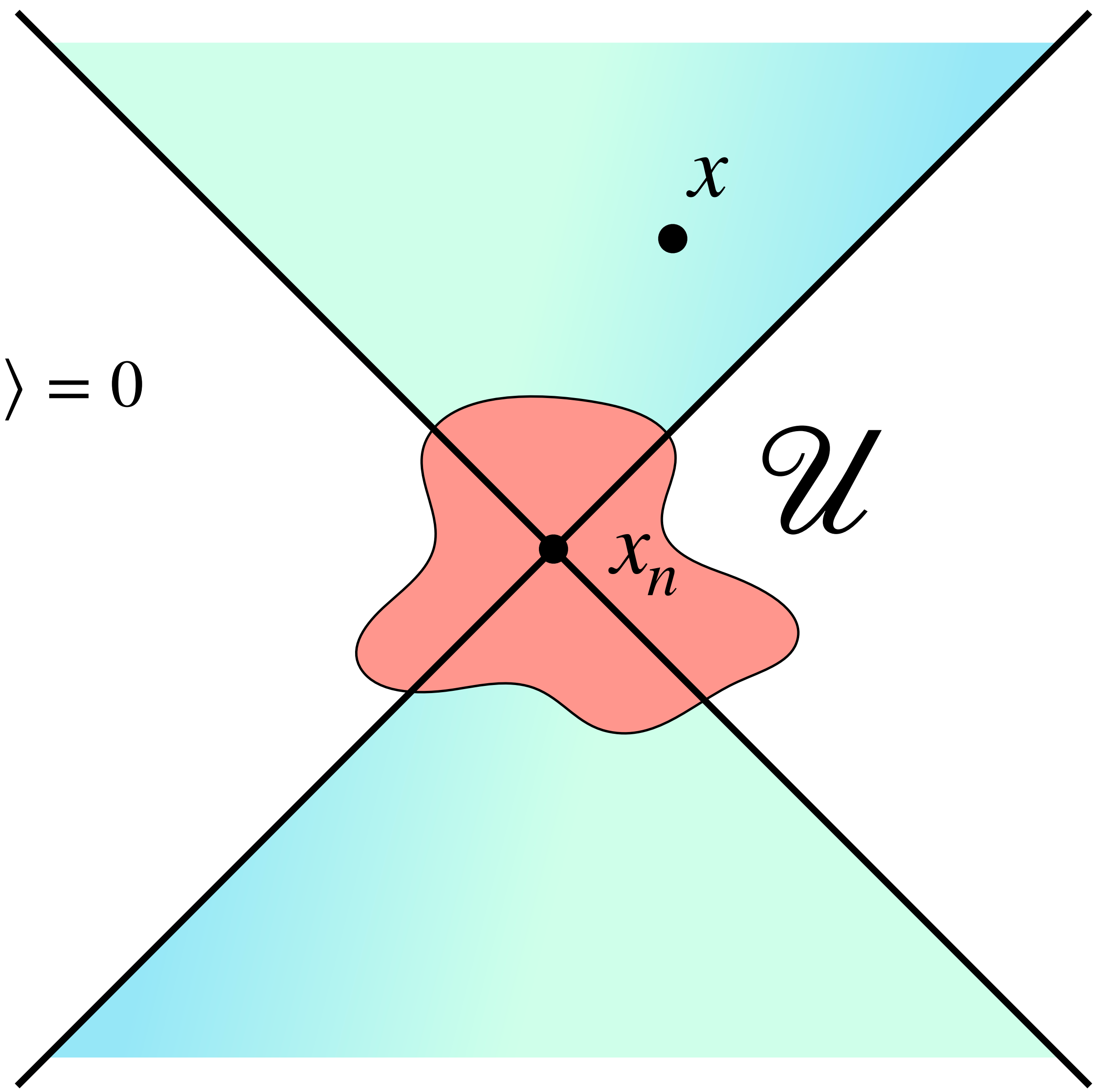

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- Any point in $\mathscr{M}_D$ can be reached by a zigzag timelike path start from x_n.

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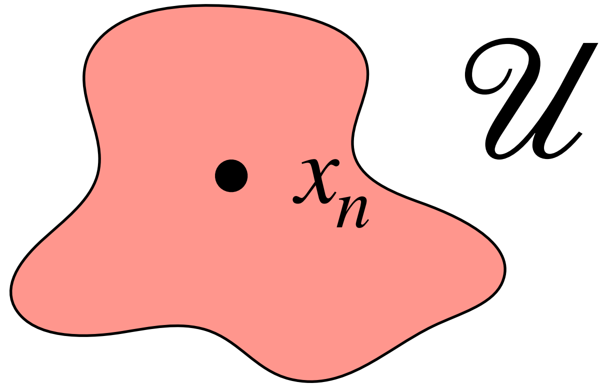

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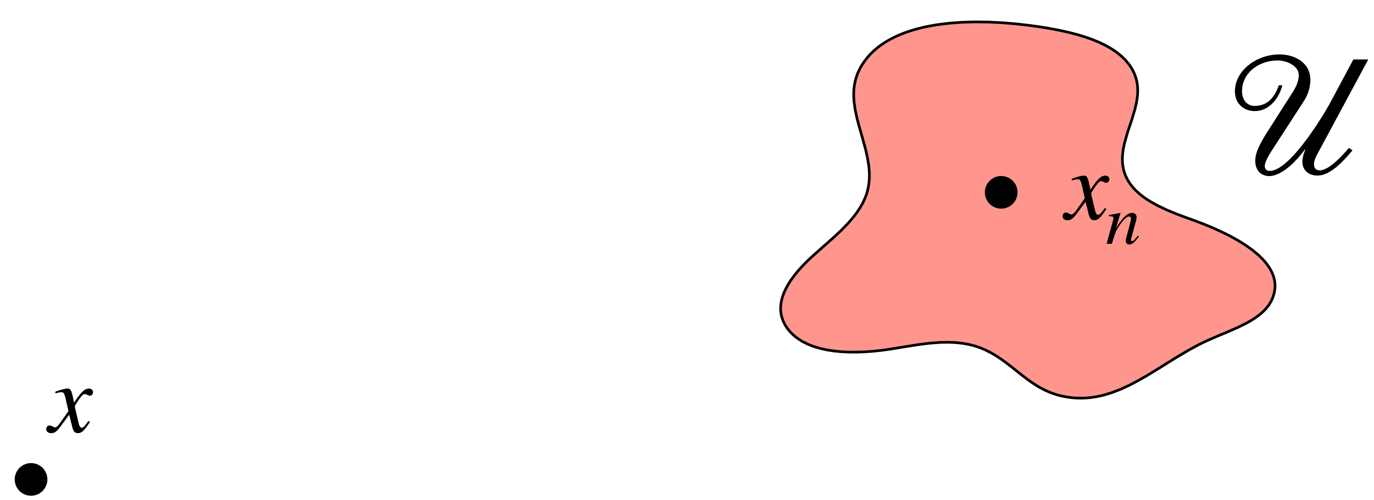

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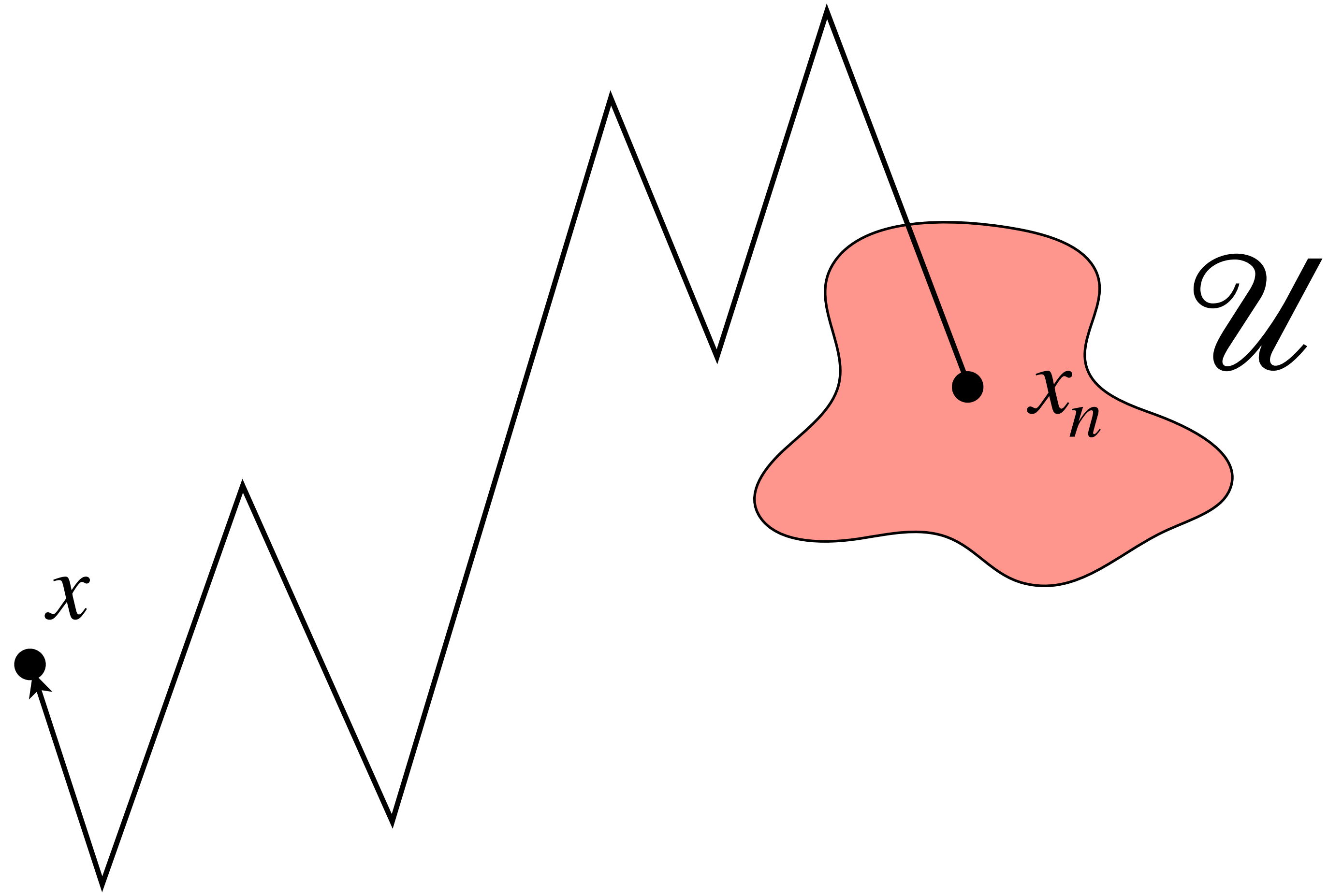

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g_2(u) &\equiv \langle \chi | \phi(x_1)\cdots\phi(x_{n-1} + u\mathbf{t})\phi(x_n + u\mathbf{t}) | \Omega \rangle \\
&= \langle \chi | \phi(x_1)\cdots\phi(x_{n-2})e^{iHu}\phi(x_{n-1})e^{-iHu}e^{iHu}\phi(x_n)e^{-iHu} | \Omega \rangle \\
&= \langle \chi | \phi(x_1)\cdots\phi(x_{n-2})e^{iHu}\phi(x_{n-1})\phi(x_n) | \Omega \rangle
\end{aligned}
$$

- So when $x_1, \cdots, x_n \in \mathscr{U}$, $\langle \chi | \phi(x_1)\cdots\phi(x_{n-2})\phi(x)\phi(x + x_n - x_{n-1}) | \Omega \rangle = 0$ for any $x \in \mathscr{M}_D$.

- Use the same method, we can first push x_{n-1} (with x_n together) to everywhere in the Minkowski spacetime, and then use the method again pushing x_n alone to any point in the Minkowski spacetime.

- So when $x_1, \cdots, x_{n-2} \in \mathscr{U}$, $\langle \chi | \phi(x_1)\cdots\phi(x_{n-2})\phi(x)\phi(y) | \Omega \rangle = 0$ for any $x, y \in \mathscr{M}_D$.

THE REEH-SCHLIEDER THEOREM

II. Proof

- Proof with Wightman function $\mathscr{W}(x_1, x_2, \cdots, x_n) \equiv \langle \Omega \,|\, \phi(x_1)\phi(x_2)\cdots\phi(x_n) \,|\, \Omega \rangle$

- There are holomorphic functions $\mathbf{W}(\zeta_1, \cdots, \zeta_{n-1})$ in $\{(\zeta_1, \cdots, \zeta_{n-1}) \,|\, \zeta_j = \xi_j - i\eta_j,\ \xi_j \in \mathbb{R}^4, \eta_j \in \mathbf{V}_+\}$, which gives

$$W(\xi_1, \xi_2, \cdots, \xi_{n-1}) = \lim_{\eta_1, \cdots, \eta_{n-1} \to 0} \mathbf{W}(\xi_1 - i\eta_1, \xi_2 - i\eta_2, \cdots, \xi_{n-1} - i\eta_{n-1})$$

$$\mathscr{W}(x_1, x_2, \cdots, x_n) = W(x_1 - x_2, x_2 - x_3, \cdots, x_n - x_{n-1})$$

- Edge of the Wedge Theorem (“契边定理”)

- With the Edge of the Wedge Theorem, see *“PCT, Spin and Statistics, and All That”* by R. F. Streater and A. S. Wightman, or the Chinese translation 《PCT，自旋统计及其他》.

THE REEH-SCHLIEDER THEOREM

III. Vectors of bounded energy momentum

- Why vacuum? Only vacuum?

$$e^{-iHu}|\Omega\rangle = |\Omega\rangle$$

- Invariance condition is too strong.

- Holomorphic in u is enough for the proof.

- This is not a generic property for arbitrary states in $\mathscr{H}$, because the operator $\exp(-ic^{\mu}P_{\mu})$ is no longer unitary but unbounded when the spacetime D-vector c^{μ} have non-vanished imaginary part.

THE REEH-SCHLIEDER THEOREM

III. Vectors of bounded energy momentum

- A little about the unbounded operator
- An example: one dimensional harmonic oscillator

THE REEH-SCHLIEDER THEOREM

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$$|\psi\rangle \equiv \frac{\sqrt{6}}{\pi} \sum_{n=0}^{\infty} \frac{|n\rangle}{n+1}$$

THE REEH-SCHLIEDER THEOREM

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THE REEH-SCHLIEDER THEOREM

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$$H|\psi\rangle = \frac{\sqrt{6}}{\pi}\sum_{n=0}^{\infty}\frac{H|n\rangle}{n+1} = \frac{\sqrt{6}}{\pi}\sum_{n=0}^{\infty}\frac{\hbar\omega}{n+1}\left(n+\frac{1}{2}\right)|n\rangle$$

THE REEH-SCHLIEDER THEOREM

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- The state $H|\psi\rangle$ is not normalizable
- Generic property: an unbounded operator can not be defined on the whole Hilbert space.

THE REEH-SCHLIEDER THEOREM

III. Vectors of bounded energy momentum

- A little about the unbounded operator

- Eigenvalue problem for unbounded self-adjoint operator: spectral theorem

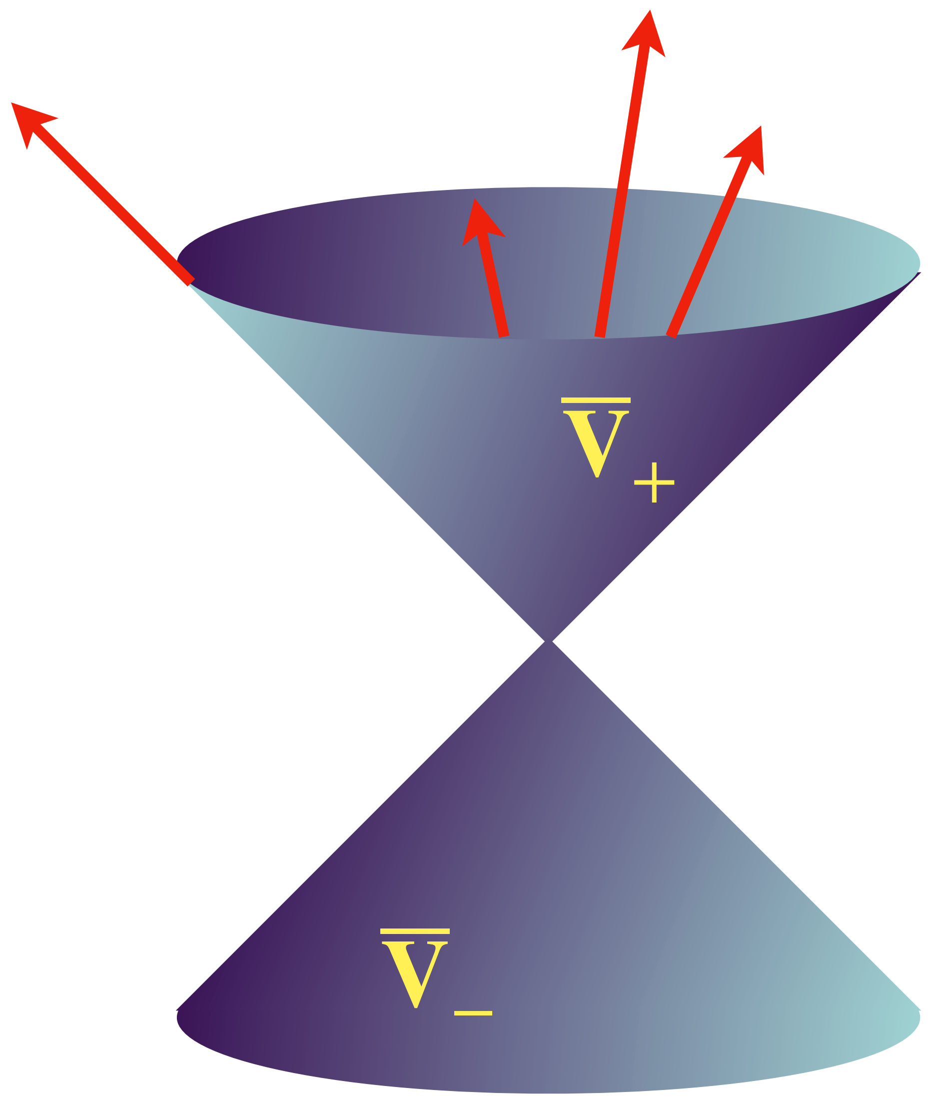
$$\hat{A} = \int_{\sigma(A)} \lambda \ \widehat{d\Pi_\lambda}$$

THE REEH-SCHLIEDER THEOREM

III. Vectors of bounded energy momentum

- As unbounded self-adjoint operator(s), the generators of the translation action P_μ have spectral in the future lightcone $\overline{\mathbf{V}}_+$!

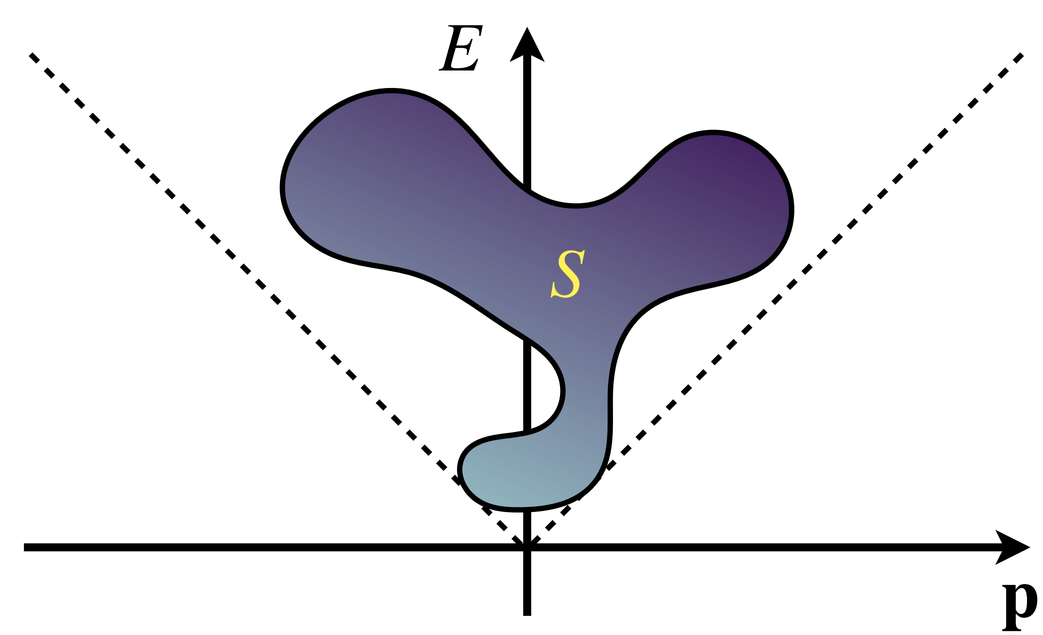
- This is an assumption of QFT.



THE REEH-SCHLIEDER THEOREM

III. Vectors of bounded energy momentum

- For compact (so bounded) region S in $\overline{\mathbf{V}}_+$, the state $\Pi_S|\Psi\rangle$ is a state on which translation group acts holomorphically.
- Any state $|\Psi\rangle$ can be approximated by a sequence $\{\Pi_{S_i}|\Psi\rangle\}$ in which each state can be used instead of the vacuum state in the Reeh-Schlieder theorem.
- These states can be used to get the Reeh-Schlieder theorem in non-vacuum superselection sector.



THE REEH-SCHLIEDER THEOREM

III. Vectors of bounded energy momentum

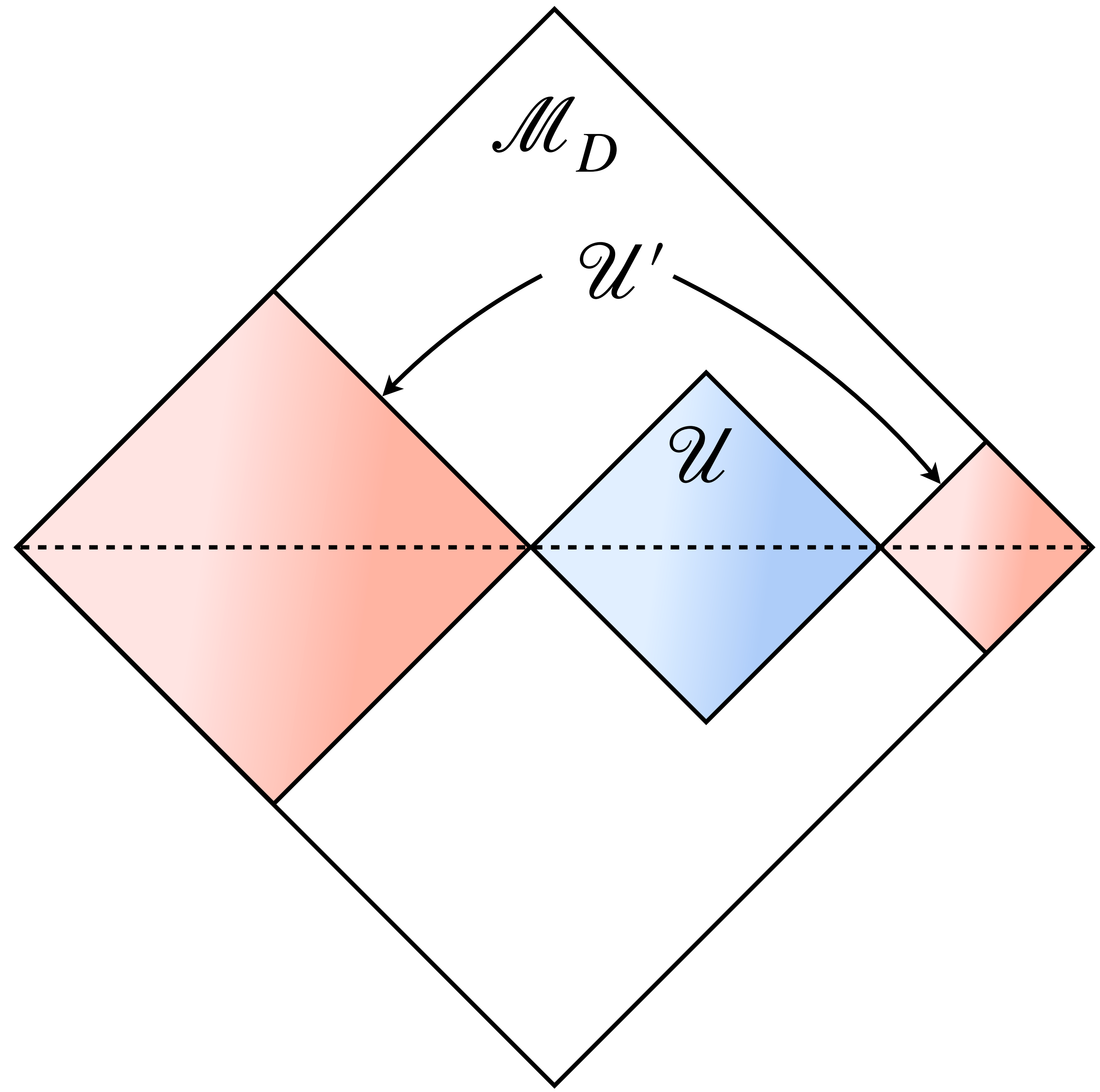
- It is non-trivial to generalize the concept of vacuum and the Reeh-Schlieder theorem to a QFT in generic spacetime.

- For **global hyperbolic** spacetime and anti-de Sitter spacetime, there are some results.

THE REEH-SCHLIEDER THEOREM

IV. An important corollary

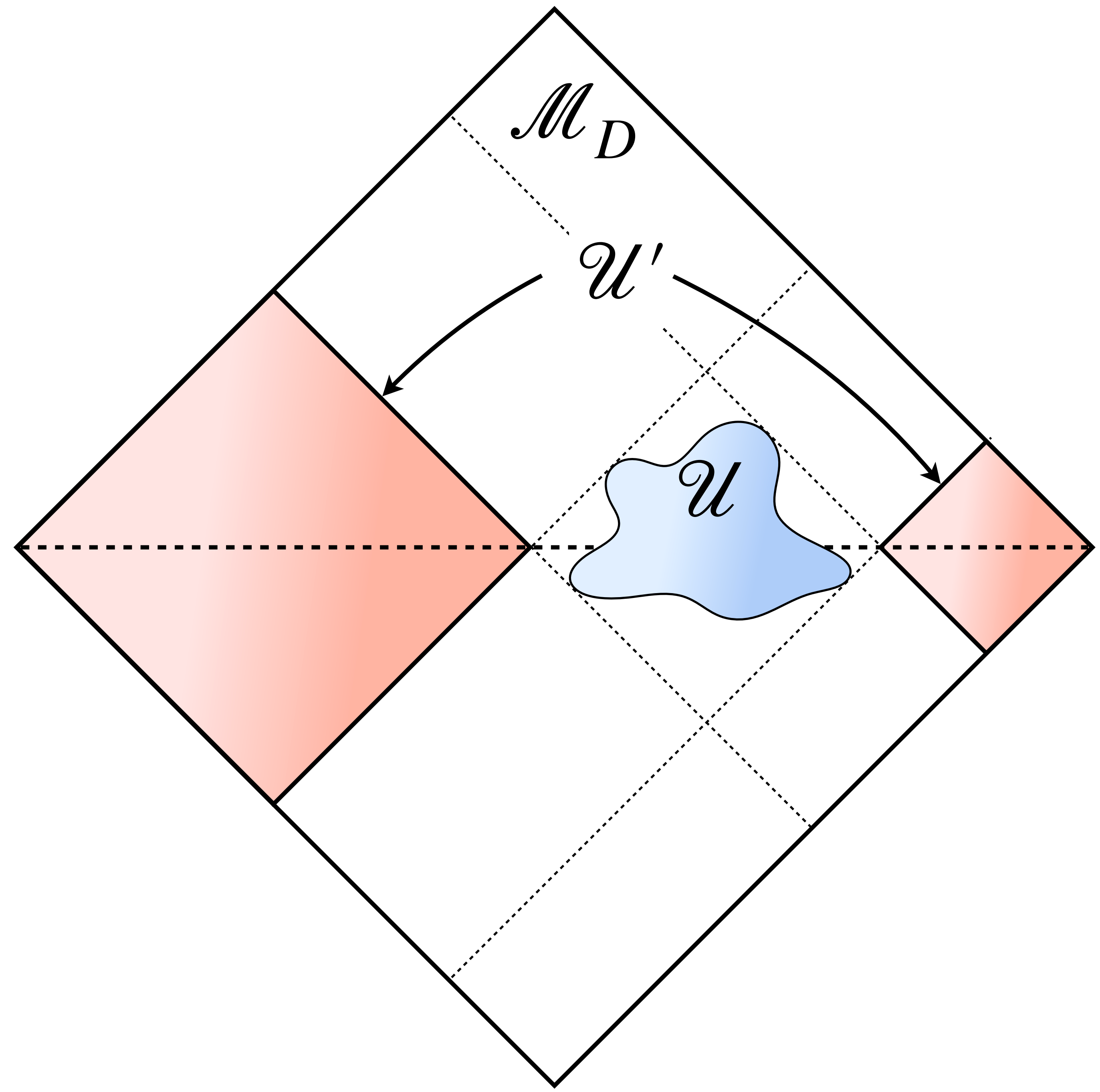
- Causal complement



THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- Causal complement
- Another example



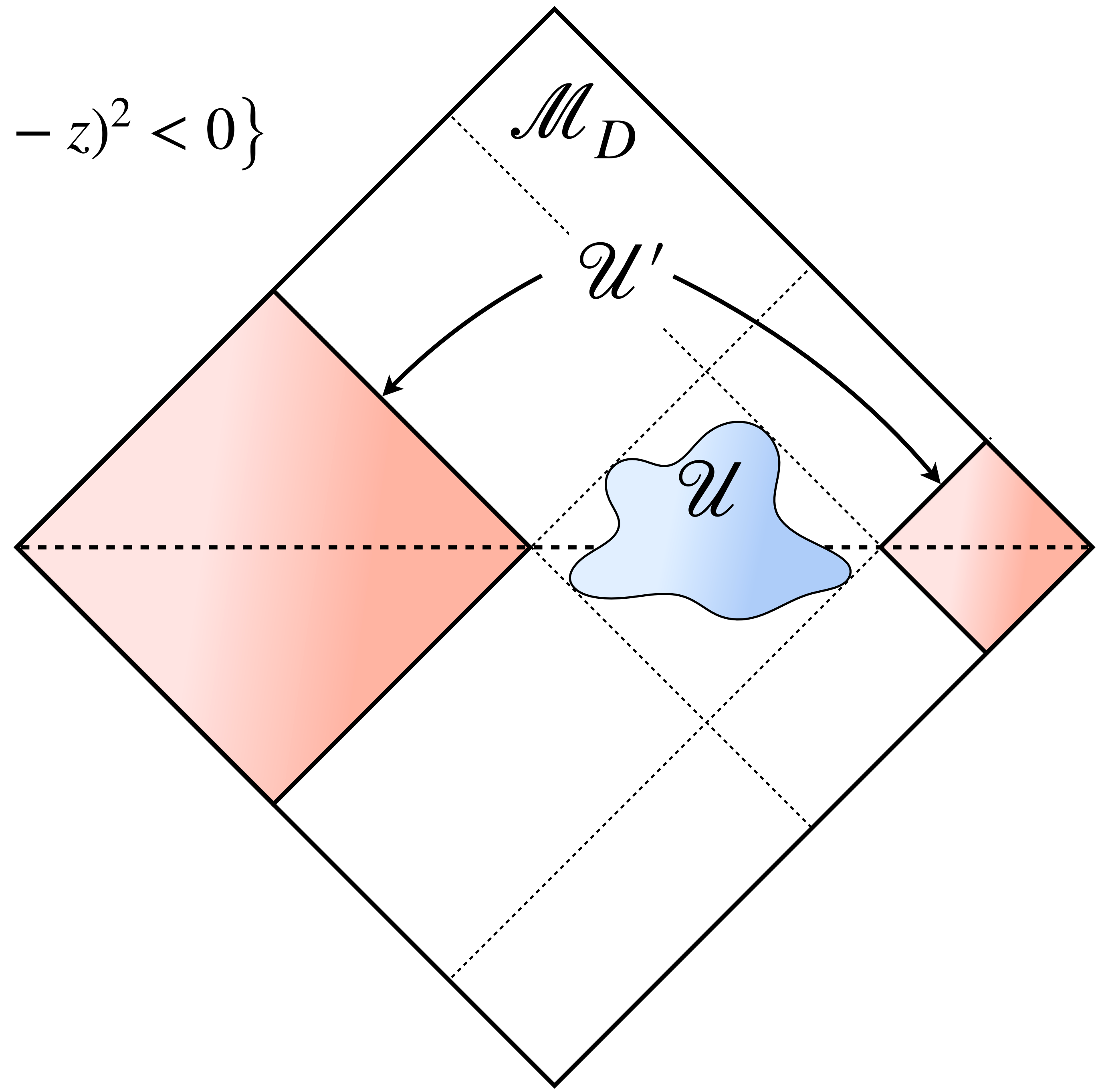
THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- Causal complement

$$\mathscr{U}' \equiv \left\{ z \in \mathscr{M}_D \,|\, \forall\, x \in \mathscr{U},\, (x-z)^2 < 0 \right\}$$

$$\mathscr{U} \subseteq \mathscr{U}''$$

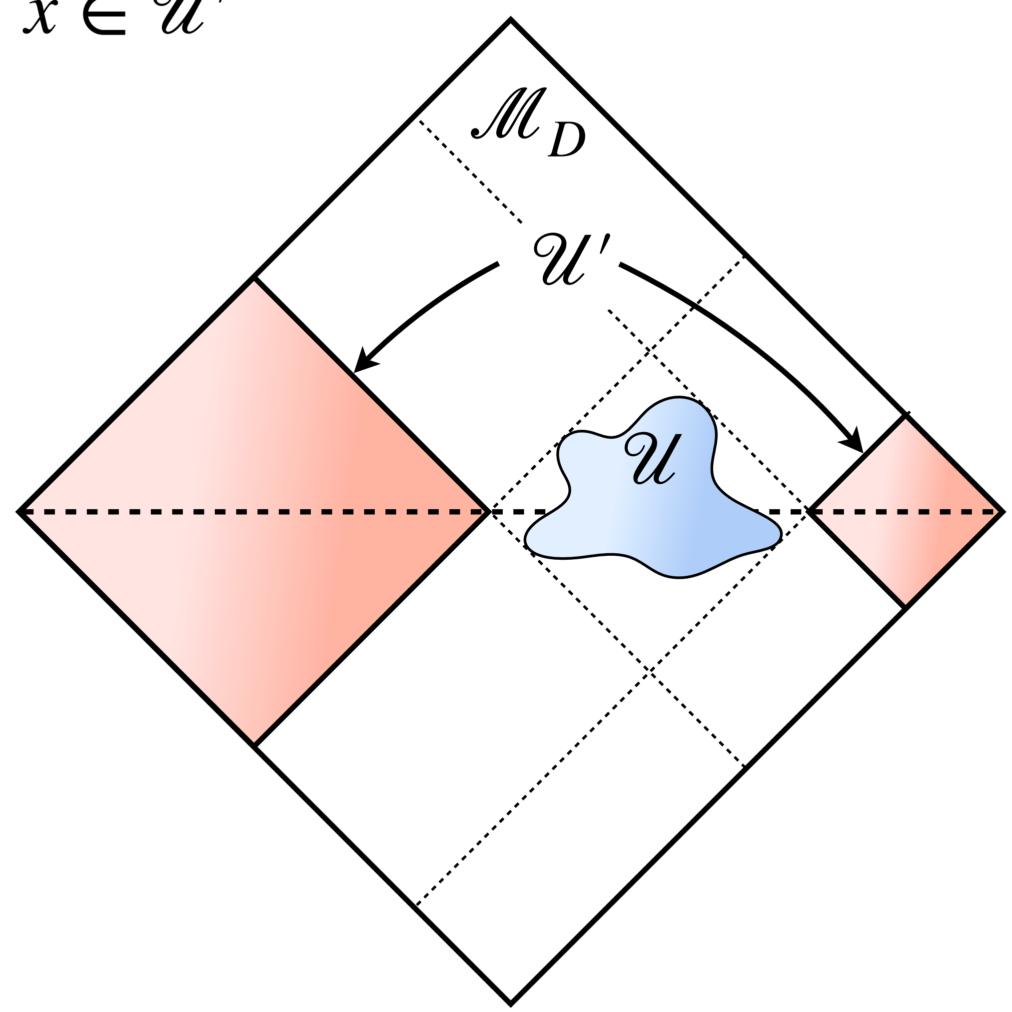


THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- Causality condition in QFT: let $\mathbf{a}$ be any operator supported in the spacetime region $\mathscr{U}$ (not necessarily constructed from a product of finitely many local operators),

$$[\phi(x), \mathbf{a}] = 0, \qquad x \in \mathscr{U}'$$

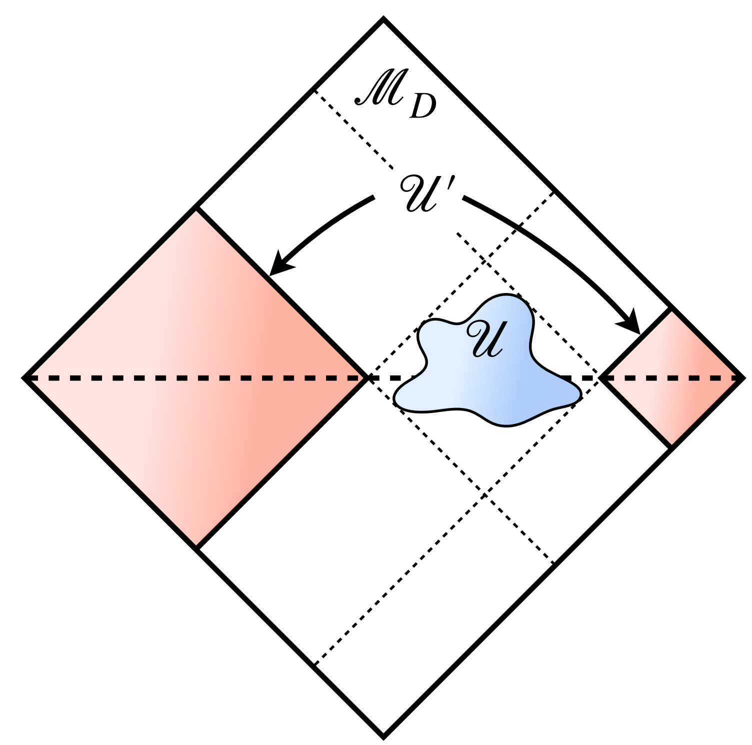


THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- If $\mathbf{a}$ is an supported in the spacetime region $\mathscr{U}$ and annihilate the vacuum, $\mathbf{a}\,|\,\Omega\rangle = 0$, then for $\forall\ x_1, \cdots, x_n \in \mathscr{U}'$

$$\mathbf{a}\ \phi(x_1)\cdots\phi(x_n)\,|\,\Omega\rangle = \phi(x_1)\cdots\phi(x_n)\ \mathbf{a}\,|\,\Omega\rangle = 0$$



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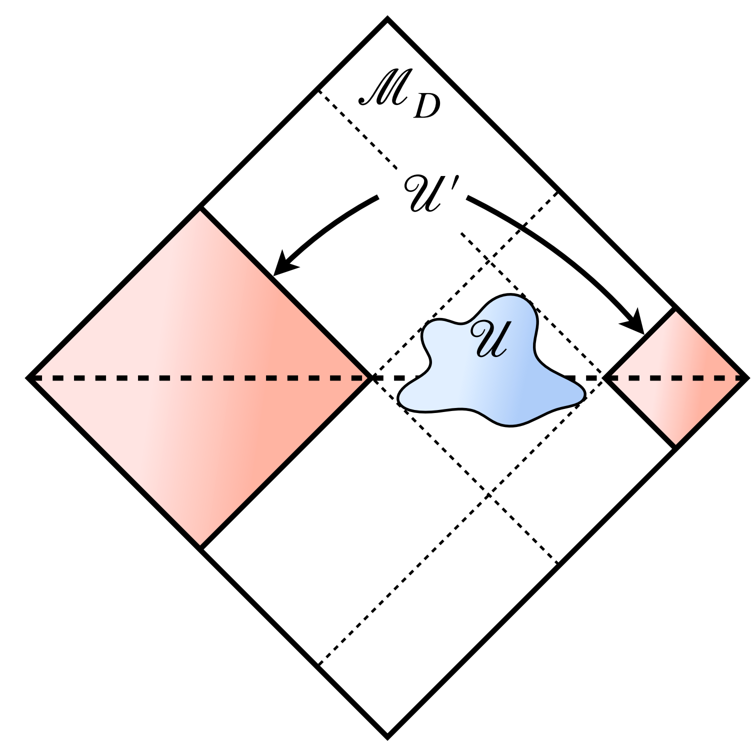
$$\mathbf{a}\ \phi(x_1)\cdots\phi(x_n)\,|\,\Omega\rangle = \phi(x_1)\cdots\phi(x_n)\ \mathbf{a}\,|\,\Omega\rangle = 0$$

- The Reeh-Schlieder theorem tells us that $\phi(x_1)\cdots\phi(x_n)\,|\,\Omega\rangle$ are dense in the vacuum sector.

- Corollary: $\mathbf{a} = \mathbf{0}$ in the vacuum sector.



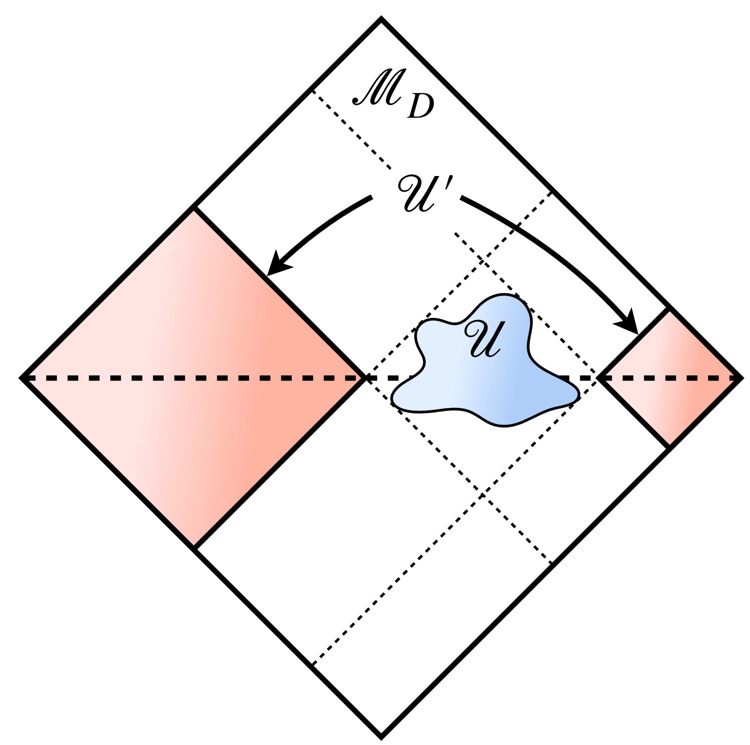
THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- "Local algebra" of quantum field theory

$$\forall\ \mathscr{U} \subset \mathscr{M}_D \ \mapsto\ \mathfrak{A}(\mathscr{U})$$

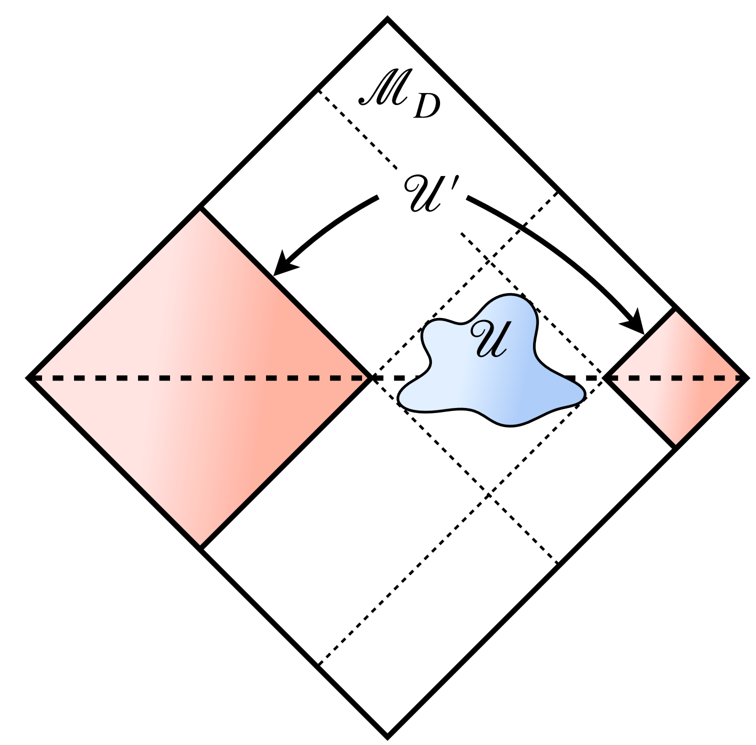
- $\mathfrak{A}(\mathscr{U})$ is the algebra generated by the **bounded** self-adjoint operators of the observables which can be locally measured in the spacetime region $\mathscr{U}$.



THE REEH-SCHLIEDER THEOREM

IV. An important corollary

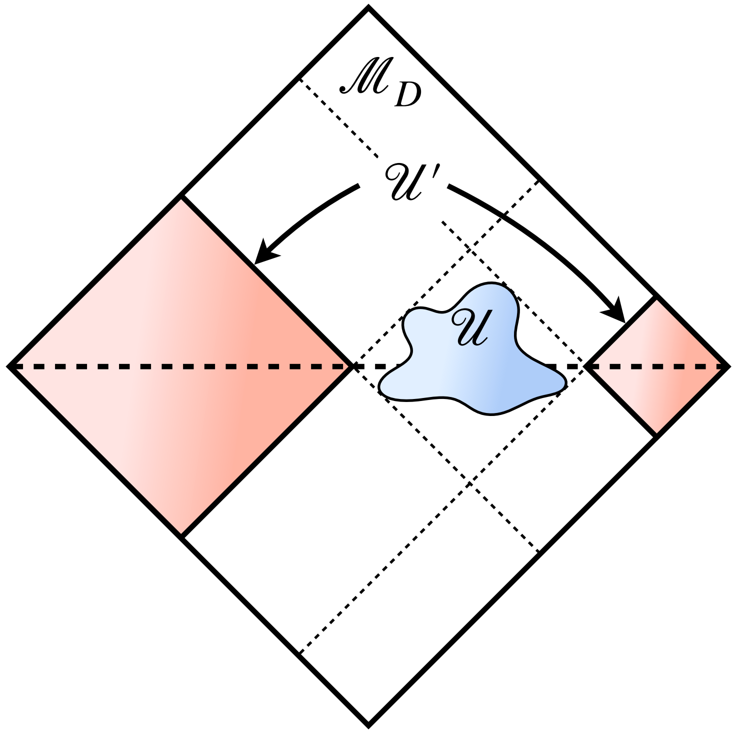
- Cyclic vector and separating vector for $\mathfrak{A}(\mathscr{U})$



THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- Cyclic vector and separating vector for $\mathfrak{A}(\mathscr{U})$
- If $\overline{\mathfrak{A}(\mathscr{U})\,|\,\Psi\rangle} = \mathscr{H}_0$, then $|\,\Psi\rangle$ is called a cyclic vector for $\mathfrak{A}(\mathscr{U})$.



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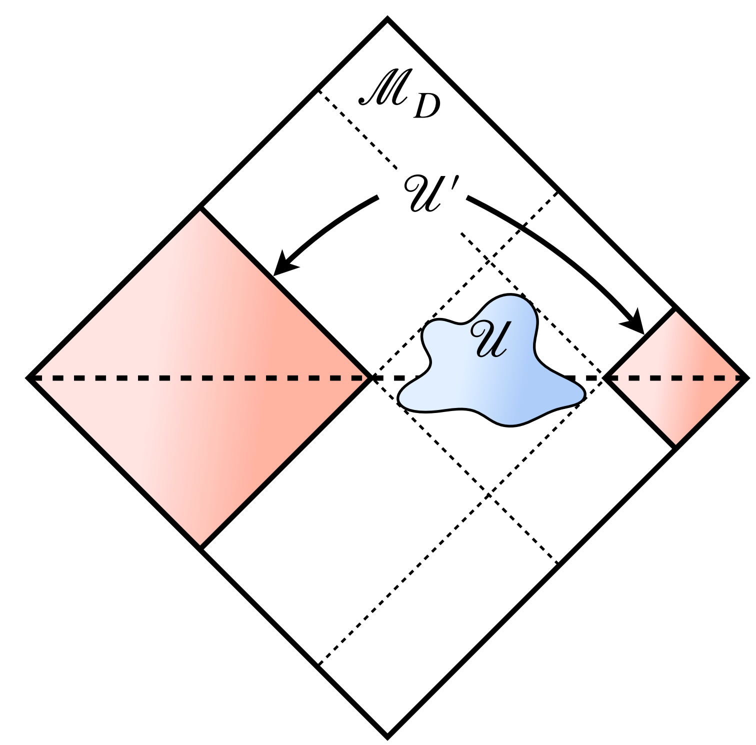
IV. An important corollary

- Cyclic vector and separating vector for $\mathfrak{A}(\mathscr{U})$
- If $\overline{\mathfrak{A}(\mathscr{U})\,|\,\Psi\rangle} = \mathscr{H}_0$, then $|\,\Psi\rangle$ is called a cyclic vector for $\mathfrak{A}(\mathscr{U})$.
- If $\mathbf{a}\,|\,\Psi\rangle = 0,\ \mathbf{a} \in \mathfrak{A}(\mathscr{U}) \ \Rightarrow\ \mathbf{a} = \mathbf{0}$, then $|\,\Psi\rangle$ is called a separating vector for $\mathfrak{A}(\mathscr{U})$.

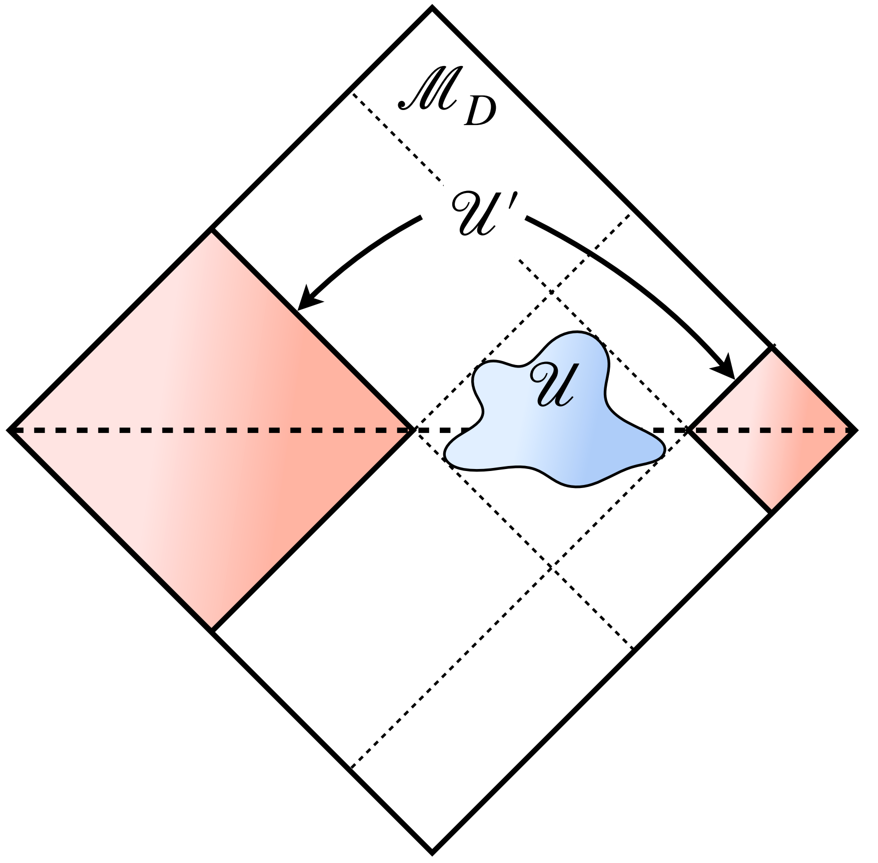


THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- If $|\Psi\rangle$ is a cyclic vector for $\mathfrak{A}(\mathscr{U})$, and $\mathbf{a}' \in \mathfrak{A}(\mathscr{U}')$ vanishes $|\Psi\rangle$, then because $[\mathbf{a}, \mathbf{a}'] = 0$, $\mathbf{a}'$ vanishes a dense subset of $\mathscr{H}_0$, so it must be $\mathbf{0}$.

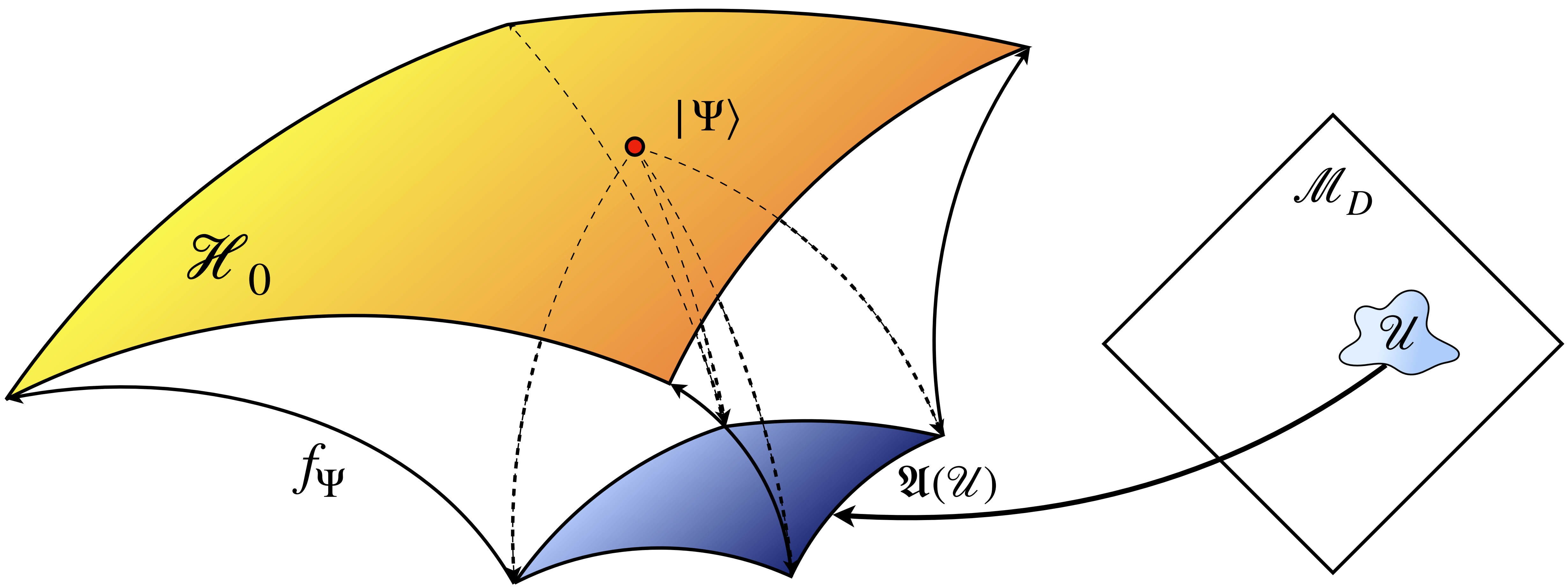
- $\Rightarrow$ A cyclic vector for $\mathfrak{A}(\mathscr{U})$ is a separating vector for $\mathfrak{A}(\mathscr{U}')$.



THE REEH-SCHLIEDER THEOREM

IV. An important corollary

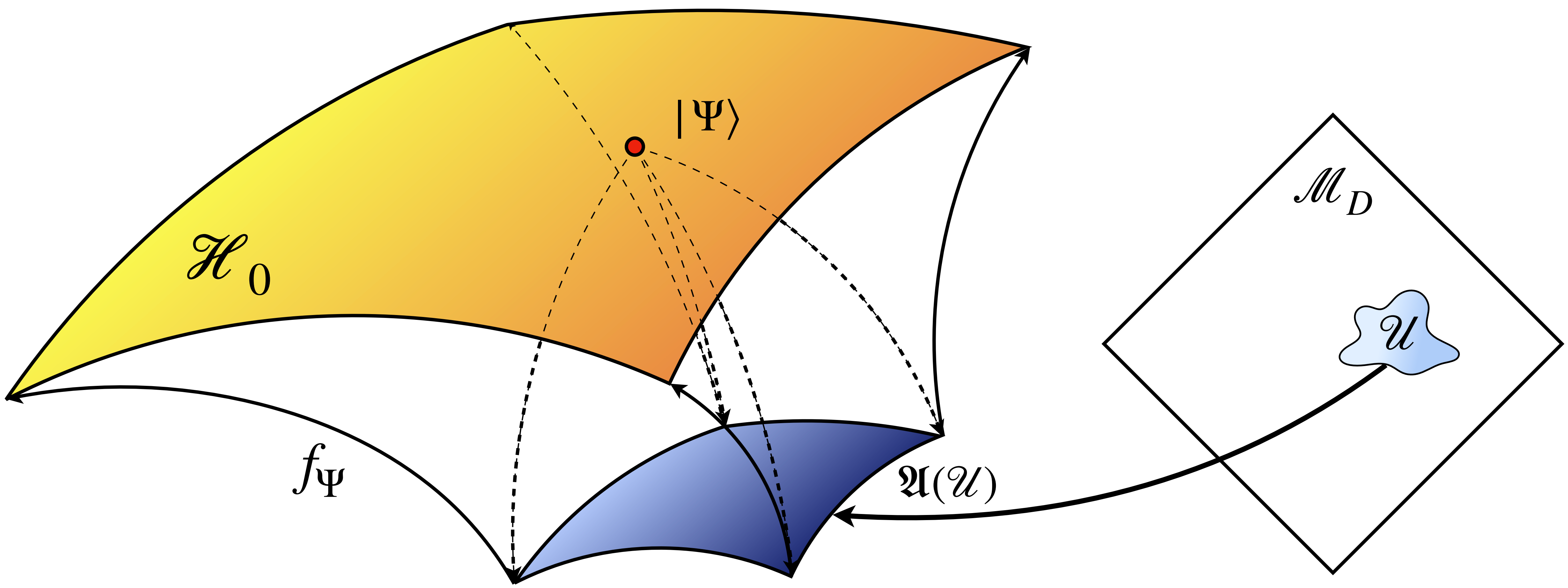
- State—operator "corresponding": every $|\Psi\rangle$ defines a map from $\mathfrak{A}(\mathscr{U})$ to $\mathscr{H}_0$ by f_Ψ : $\mathbf{a} \in \mathfrak{A}(\mathscr{U}) \mapsto \mathbf{a}|\Psi\rangle \in \mathscr{H}_0$.



THE REEH-SCHLIEDER THEOREM

IV. An important corollary

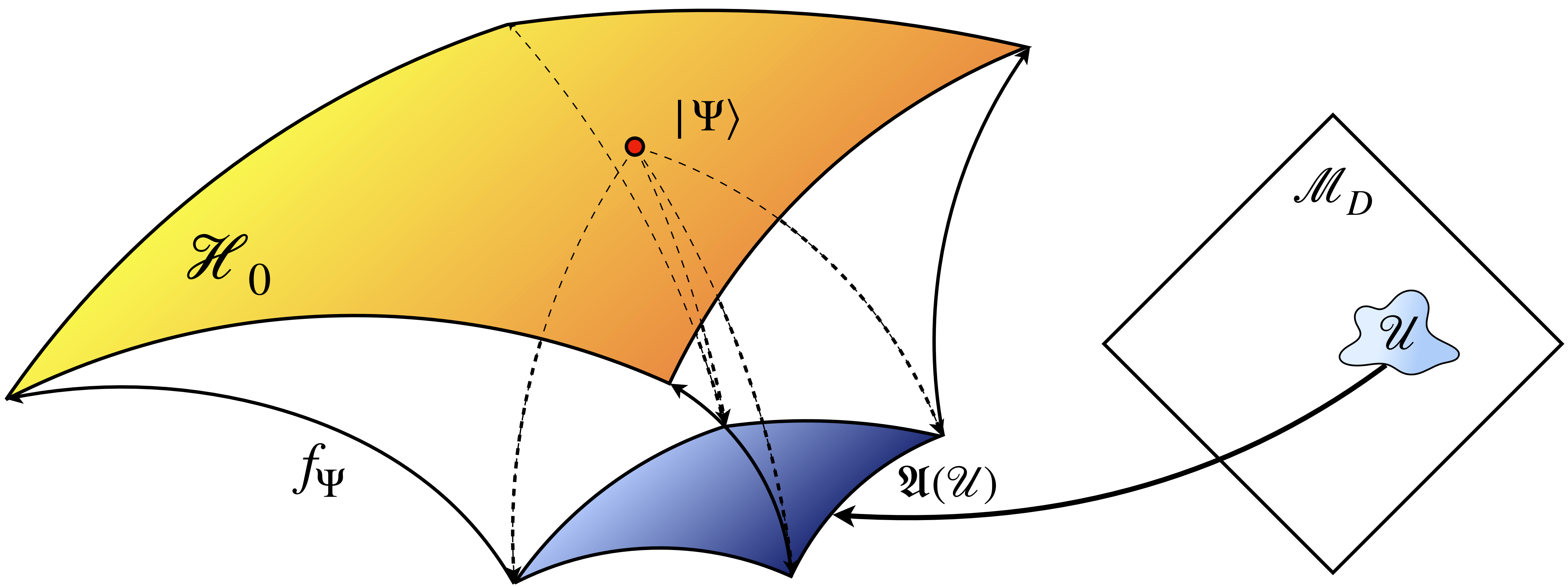
- If $\overline{\text{im}(f_\Psi)} = \mathscr{H}_0$, then $|\Psi\rangle$ is a cyclic vector for $\mathfrak{A}(\mathscr{U})$.
- If $\ker(f_\Psi) = \mathbf{0}$, then $|\Psi\rangle$ is a separating vector for $\mathfrak{A}(\mathscr{U})$.



THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- Reeh-Schlieder theorem $\Rightarrow$ The vacuum state $|\Omega\rangle$ is both cyclic and separating vector for local algebra $\mathfrak{A}(\mathscr{U})$ and $\mathfrak{A}(\mathscr{U}')$ on any open subset $\mathscr{U}$ of the Minkowski spacetime.



THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- An example of non-cyclic vector: (complex fermion field)
- For $f \in C^\infty(\mathscr{M}_D)$, $\mathbf{supp}\, f \subset \mathscr{U}$, define $\psi_f \equiv \int d^D x\, f(x)\psi(x)$.
- Because $\psi_f^2 = \mathbf{0}$, we have $\psi_f\ (\psi_f|\chi\rangle) = 0$ for any $|\chi\rangle$.
- So for any $|\chi\rangle$, $\psi_f|\chi\rangle$ can not be separating vector for $\mathscr{U}$.

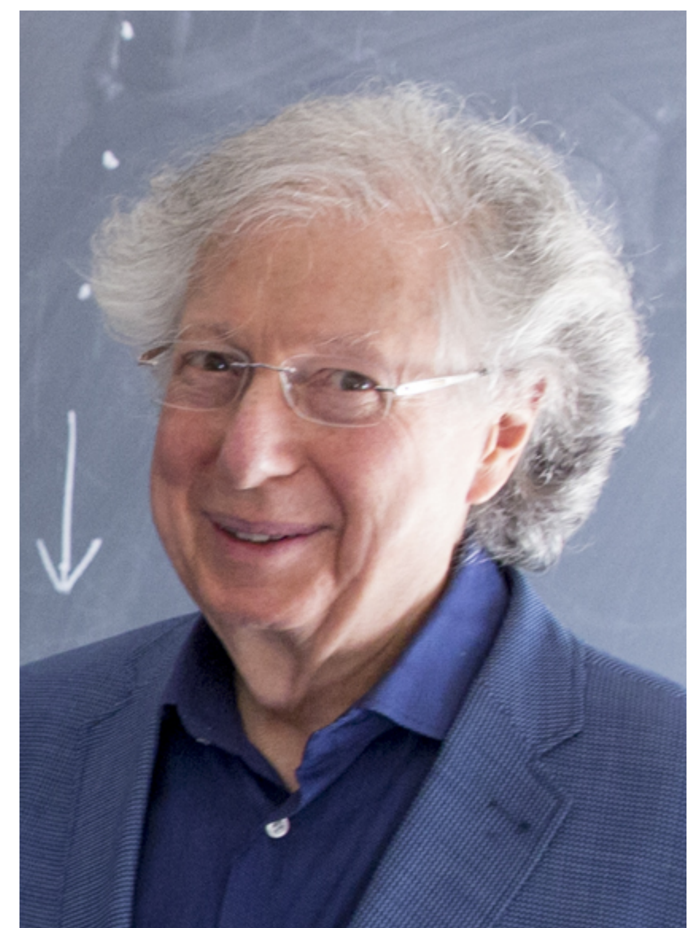
THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- An important corollary: "*Nonpositivity of the Energy Density in Quantized Field Theories*", H. Epstein, V. Glaser, A. Jaffe, Nuovo Cimento 36 (1965) 1016–1022.

Henri Epstein

Vladimir Jurko Glaser
(1924/04/21-1984/01/22)



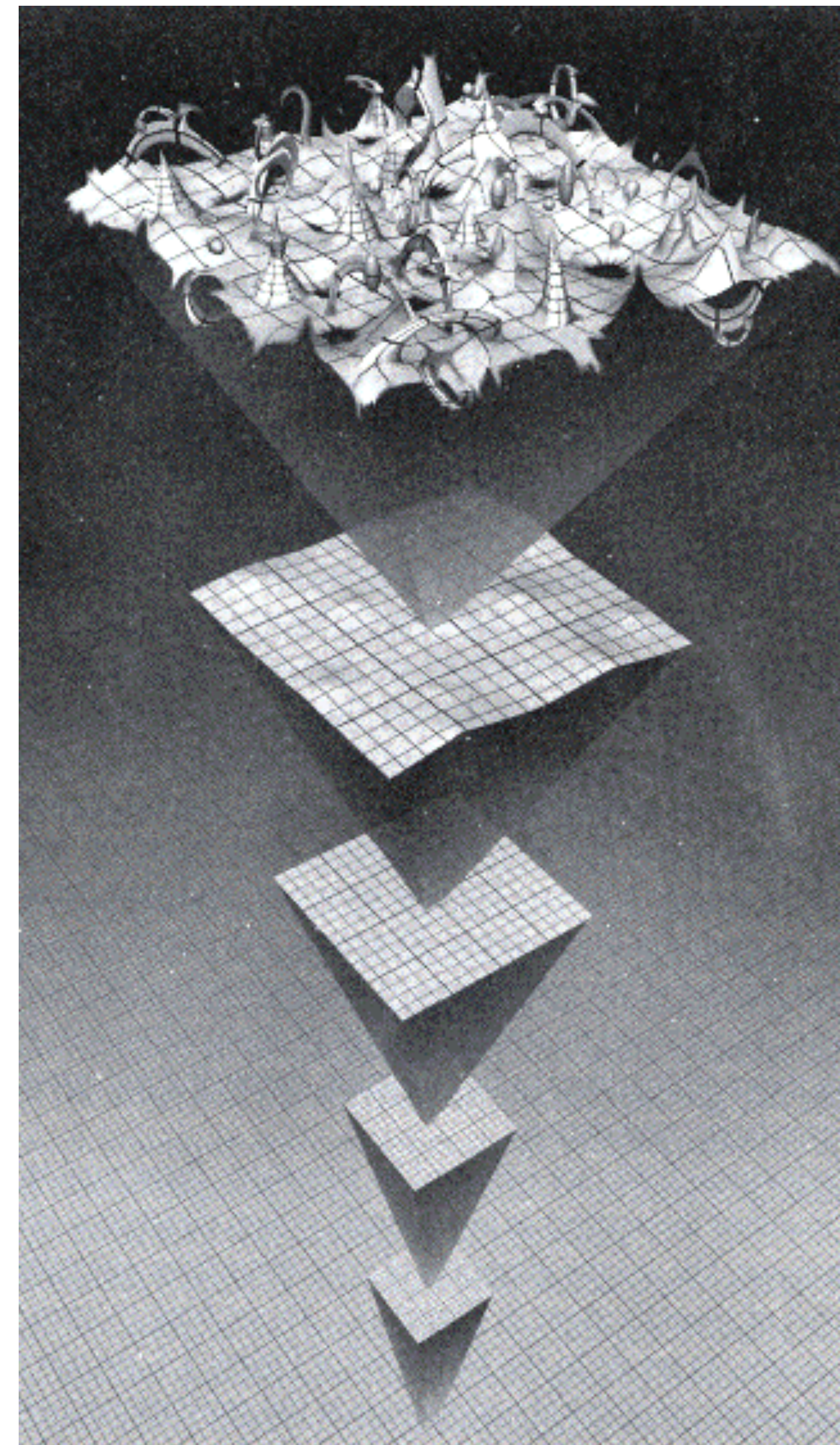
Arthur Michael Jaffe
(1937/12/22-)

THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- An important corollary:
- Local observable — Energy momentum tensor $T^{\mu\nu}(x)$
- If T^{00}_f ($f \in C^\infty(\mathscr{M}_D)$, $\mathbf{supp}\, f \subset \mathscr{U}$) vanishes vacuum, then (because vacuum is a separating vector) $T^{00}_f = 0$.
- That is not the case.
- So there must be some state $|\chi\rangle$ in $\mathscr{H}_0$,

$$\langle \chi | T^{00}_f | \Omega \rangle \neq 0$$



THE REEH-SCHLIEDER THEOREM

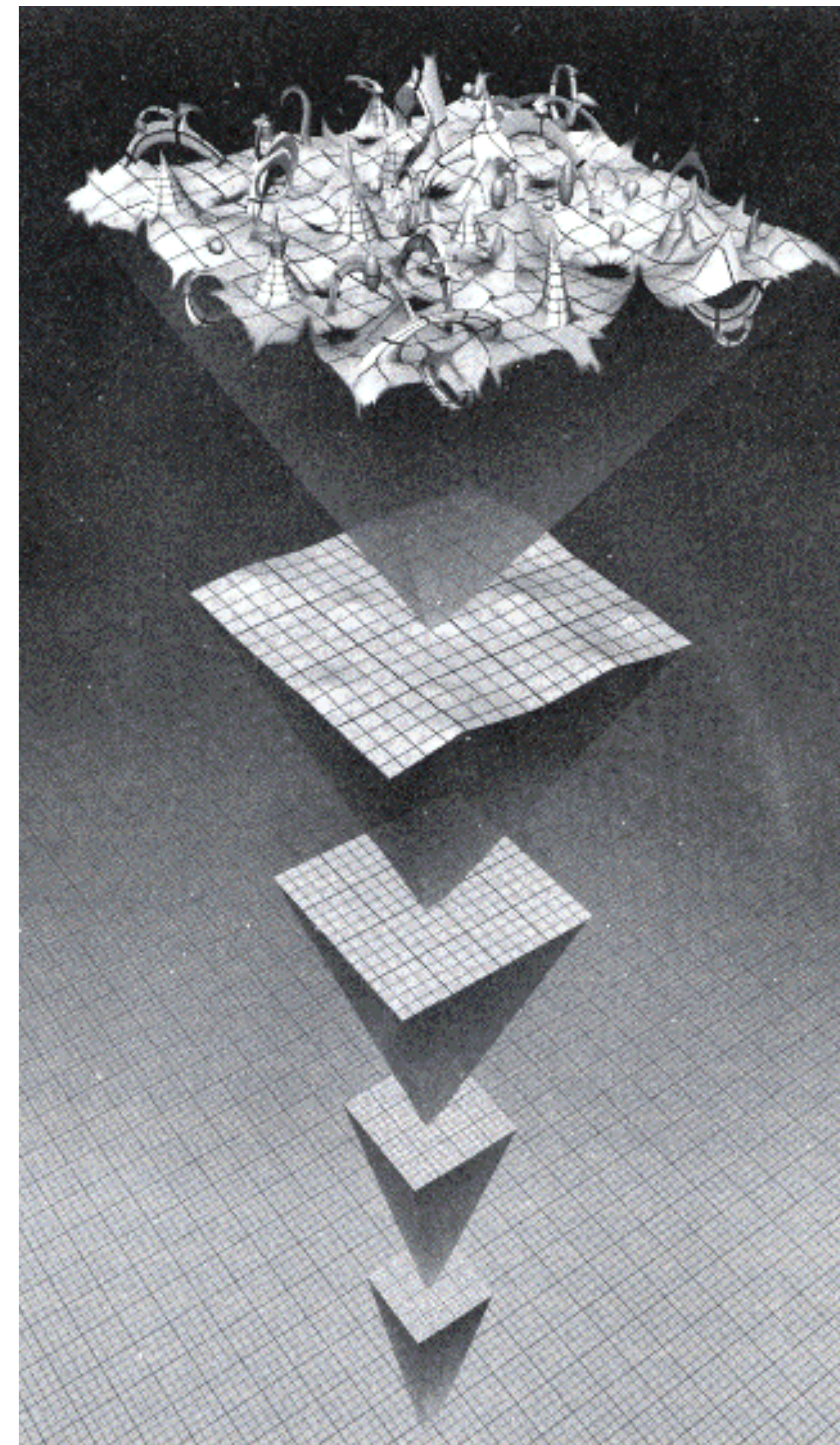
IV. An important corollary

- An important corollary:
- But the translation symmetry tells us

$$\langle\Omega\,|\,T_f^{00}\,|\,\Omega\rangle = 0$$

- So the matrix elements of operator T_f^{00} in the 2D subspace spanned by $\{\,|\chi\rangle, |\Omega\rangle\}$ are

$$\begin{pmatrix} 0 & b \\ b* & 2d \end{pmatrix}$$



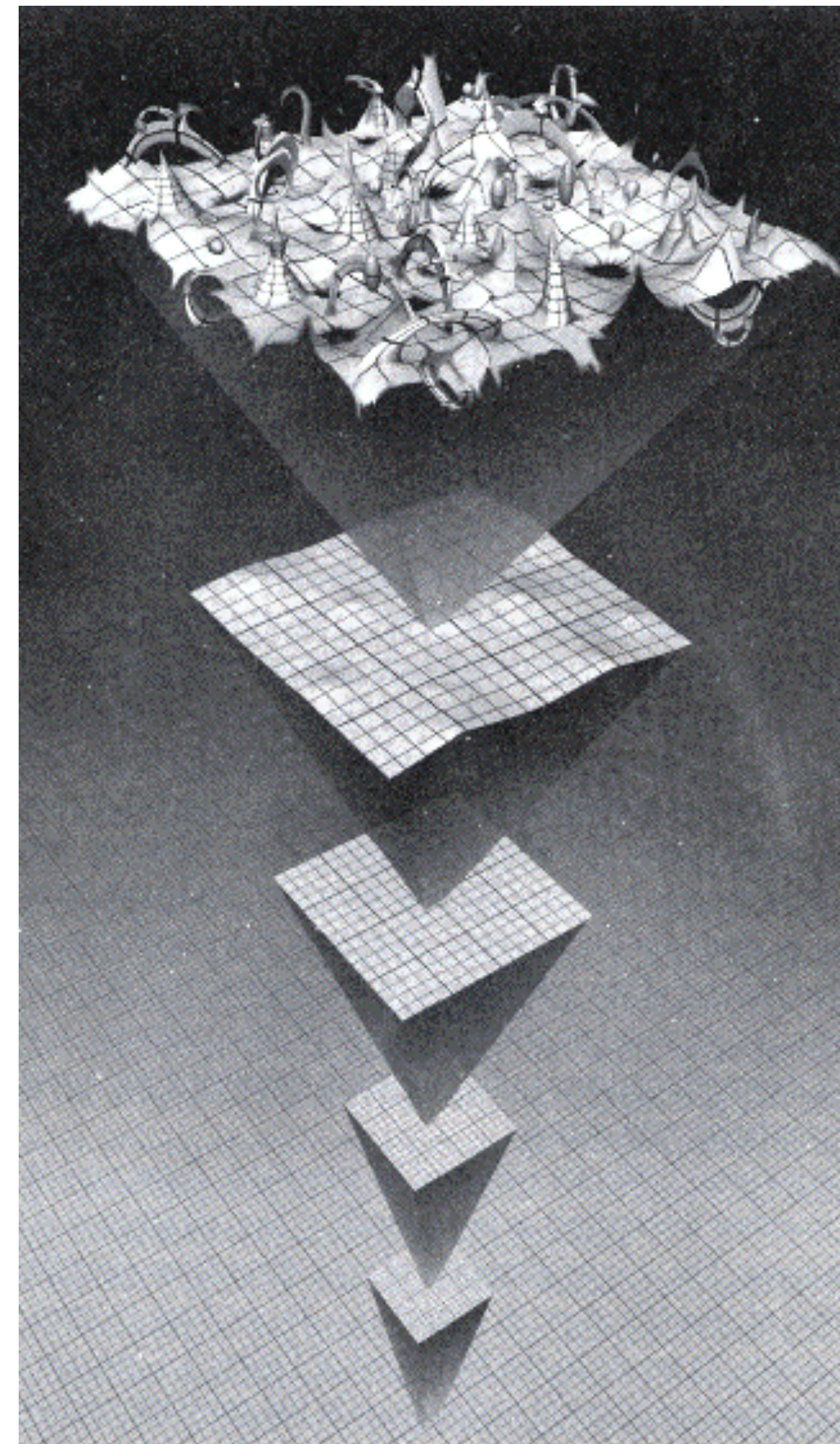
THE REEH-SCHLIEDER THEOREM

IV. An important corollary

- An important corollary:
- It is easy to check

$$T_f^{00}|\psi\rangle = \left(d - \sqrt{d^2 + |b|^2}\right)|\psi\rangle$$

$$|\psi\rangle \propto b^*|\chi\rangle - \left(d + \sqrt{d^2 + |b|^2}\right)|\Omega\rangle$$



THE REEH-SCHLIEDER THEOREM

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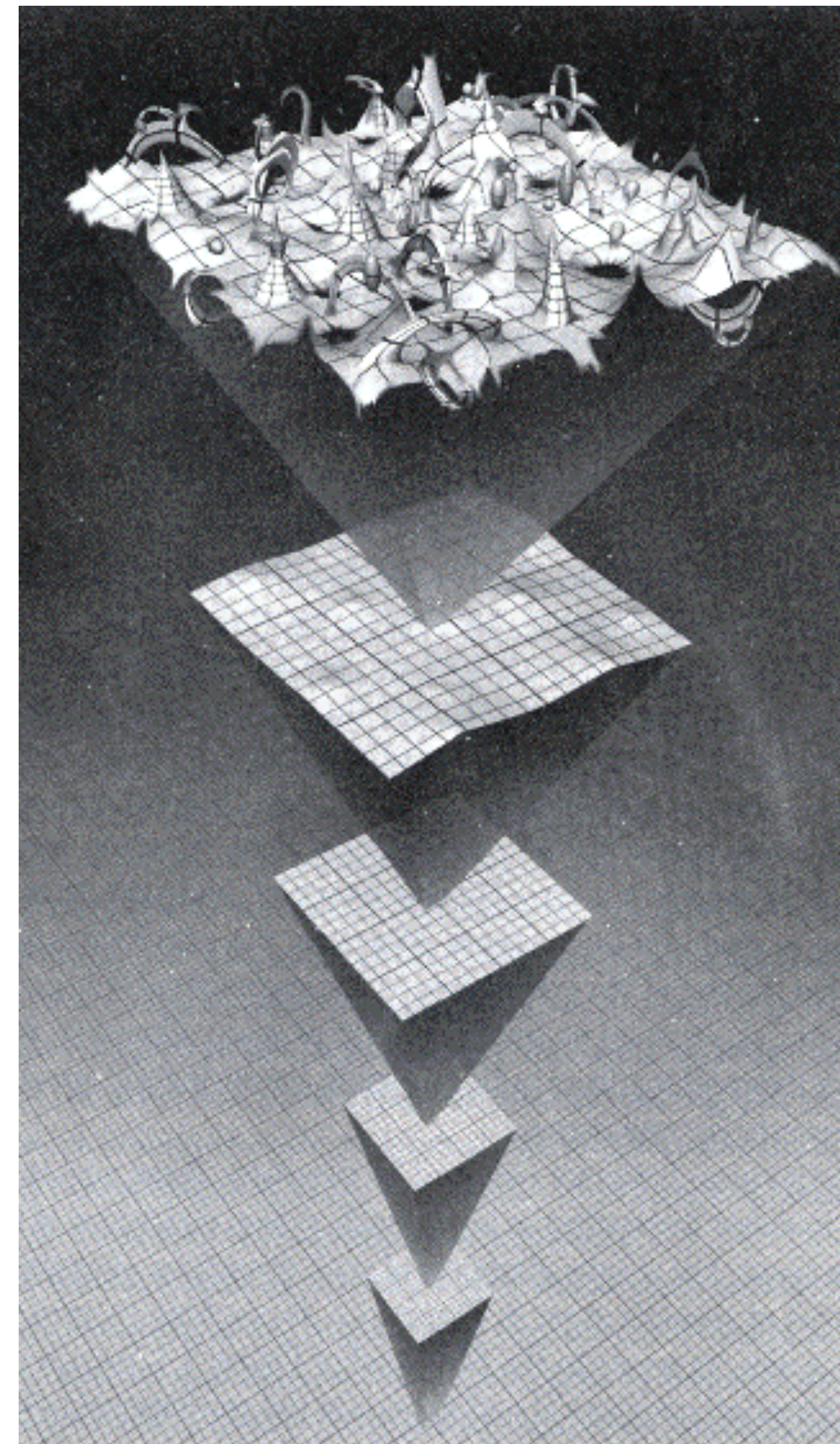
- The quantum fluctuation could lead to negative energy density in any finite spacetime region.



THE REEH-SCHLIEDER THEOREM

V. Discussion

- "Behind-the-Moon-argument"

THE REEH-SCHLIEDER THEOREM

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THE REEH-SCHLIEDER THEOREM

V. Discussion

- “Behind-the-Moon-argument”

$$\langle \Omega | \mathbf{M} | \Omega \rangle \approx 0, \quad \mathbf{M} \in \mathfrak{A}(\mathscr{U}_{Moon})$$

THE REEH-SCHLIEDER THEOREM

V. Discussion

- “Behind-the-Moon-argument”

$$\langle \Omega | \mathbf{M} | \Omega \rangle \approx 0, \quad \mathbf{M} \in \mathfrak{A}(\mathscr{U}_{Moon})$$

$$\exists\ \mathbf{a}_n \in \mathfrak{A}(\mathscr{U}_{lab}), \quad \text{s.t. } \mathbf{a}_n | \Omega \rangle \to \mathbf{M} | \Omega \rangle, \quad \mathscr{U}_{lab} \subset \mathscr{U}'_{Moon}$$

THE REEH-SCHLIEDER THEOREM

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$$\langle \Omega | \mathbf{M} | \Omega \rangle \approx 0, \quad \mathbf{M} \in \mathfrak{A}(\mathcal{U}_{Moon})$$

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$$\Rightarrow \quad \langle \Omega | \mathbf{a}_n^\dagger \mathbf{M} \mathbf{a}_n | \Omega \rangle \to 1, \quad \therefore \ \langle \Omega | \mathbf{M} \mathbf{a}_n^\dagger \mathbf{a}_n | \Omega \rangle \to 1$$

THE REEH-SCHLIEDER THEOREM

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$$\Rightarrow \quad \langle \Omega | \mathbf{a}_n^\dagger \mathbf{M} \mathbf{a}_n | \Omega \rangle \to 1, \quad \therefore \; \langle \Omega | \mathbf{M} \mathbf{a}_n^\dagger \mathbf{a}_n | \Omega \rangle \to 1$$

- The "some operator" $\mathbf{a}_n$ is **not** a unitary operator!

THE REEH-SCHLIEDER THEOREM

V. Discussion

- “Behind-the-Moon-argument”

- If one modifies the quantum system with physical operation, which is realized by Hermitian Hamiltonian and leads to unitary evolution, it is not possible to make any change in observables in a spacelike separated region.

$$\langle \Omega | \mathbf{a}_n^\dagger \mathbf{M} \mathbf{a}_n | \Omega \rangle = \langle \Omega | \mathbf{M} \mathbf{a}_n^\dagger \mathbf{a}_n | \Omega \rangle = \langle \Omega | \mathbf{M} | \Omega \rangle$$

THE REEH-SCHLIEDER THEOREM

V. Discussion

- “Behind-the-Moon-argument”
- Correlation vs causality

$$\langle \Omega \,|\, \mathbf{M}\mathbf{a}_n^\dagger \mathbf{a}_n \,|\, \Omega \rangle \neq \langle \Omega \,|\, \mathbf{M} \,|\, \Omega \rangle \langle \Omega \,|\, \mathbf{a}_n^\dagger \mathbf{a}_n \,|\, \Omega \rangle$$

THE REEH-SCHLIEDER THEOREM

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- An example in textbooks — free field propagator

THE REEH-SCHLIEDER THEOREM

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- “Behind-the-Moon-argument”
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- An example in textbooks — free field propagator

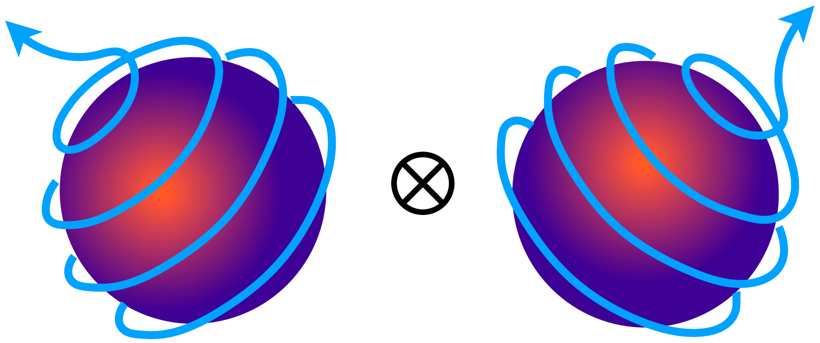
$$G_F(x,y) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^4} \int d^4p \, \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon} = \begin{cases} -\frac{1}{4\pi}\delta(s) + \frac{m}{8\pi\sqrt{s}} H_1^{(1)}(m\sqrt{s}) & s \geq 0 \\ -\frac{im}{4\pi^2\sqrt{-s}} K_1(m\sqrt{-s}) & s < 0. \end{cases}$$

THE REEH-SCHLIEDER THEOREM

V. Discussion

- Finite-dimensional quantum system

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2, \ \dim \mathscr{H}_1 = \dim \mathscr{H}_2 = n$$



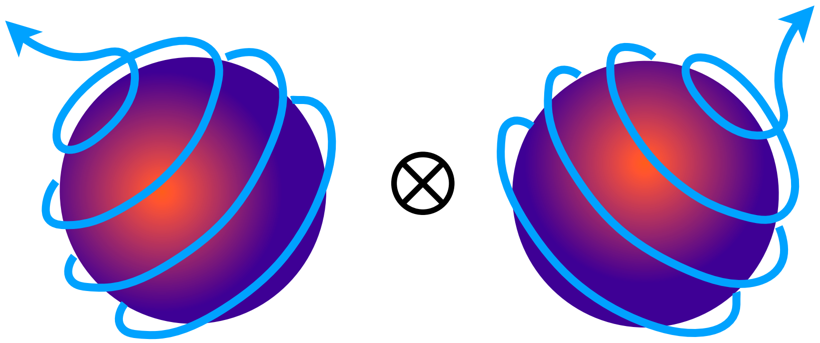
THE REEH-SCHLIEDER THEOREM

V. Discussion

- Finite-dimensional quantum system

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2, \ \dim \mathscr{H}_1 = \dim \mathscr{H}_2 = n$$

- $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are the $n \times n$ matrices acting on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.



THE REEH-SCHLIEDER THEOREM

V. Discussion

- Finite-dimensional quantum system

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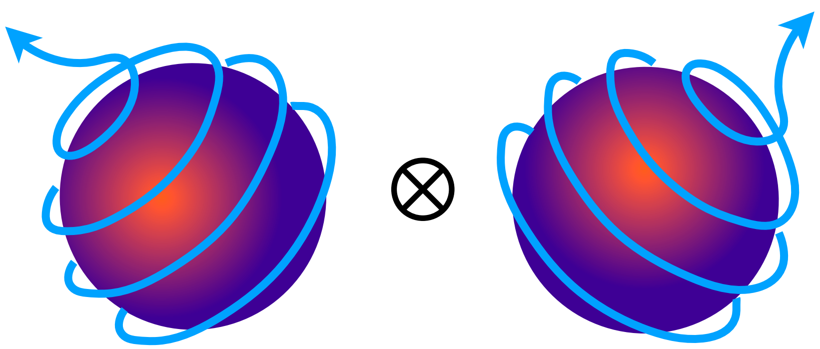
$\mathfrak{A}_1$



THE REEH-SCHLIEDER THEOREM

V. Discussion

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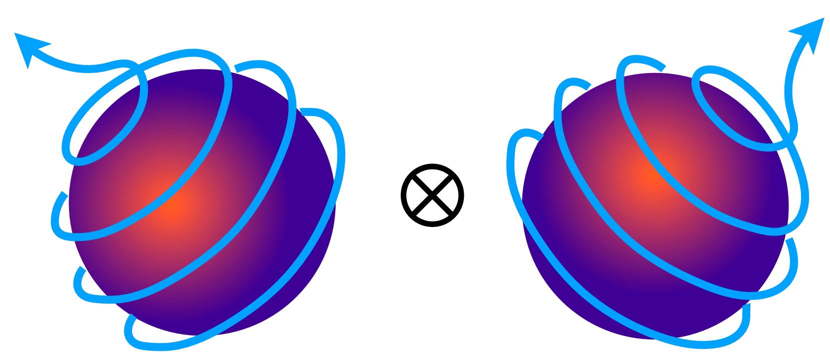
$\mathfrak{A}_2$



THE REEH-SCHLIEDER THEOREM

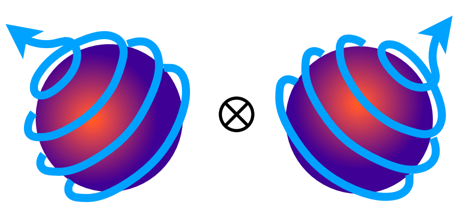
V. Discussion

- Finite-dimensional quantum system

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2, \ \dim \mathscr{H}_1 = \dim \mathscr{H}_2 = n$$

- $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are the $n \times n$ matrices acting on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

$$\Psi = \sum_{i,j=1}^{n} c_{ij} |i\rangle \otimes |j\rangle' = \mathrm{tr} \left[\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ & & \cdots & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} |1\rangle \otimes |1\rangle' & |2\rangle \otimes |1\rangle' & \cdots & |n\rangle \otimes |1\rangle' \\ |1\rangle \otimes |2\rangle' & |2\rangle \otimes |2\rangle' & \cdots & |n\rangle \otimes |2\rangle' \\ & & \cdots & \\ |1\rangle \otimes |n\rangle' & |2\rangle \otimes |n\rangle' & \cdots & |n\rangle \otimes |n\rangle' \end{pmatrix} \right]$$



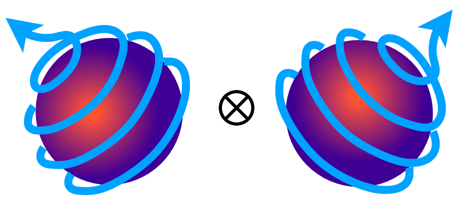
THE REEH-SCHLIEDER THEOREM

V. Discussion

- Finite-dimensional quantum system
- If $\mathscr{A} \in \mathfrak{A}_1$, $\mathscr{B} \in \mathfrak{A}_2$, and $\mathscr{A}\,|\,i\rangle = \sum_{j=1}^{n} |\,j\rangle A_{ji}$, $\mathscr{B}\,|\,i\rangle' = \sum_{j=1}^{n} |\,j\rangle' B_{ji}$, we have

$$\{\mathscr{A} \otimes \mathscr{B}\}\Psi = \sum_{i,j=1}^{n} c_{ij}(\mathscr{A}\,|\,i\rangle) \otimes (\mathscr{B}\,|\,j\rangle') = \sum_{i,j,k,\ell=1}^{n} c_{ij} A_{ki} B_{\ell j}\,|\,k\rangle \otimes |\,\ell\rangle'$$

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ & & \cdots & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \quad \to \quad ACB^T$$



THE REEH-SCHLIEDER THEOREM

V. Discussion

- Finite-dimensional quantum system

- Singular value decomposition theorem: any $m \times n$ complex matrix can be factorized as $\mathbf{U\Sigma V}^*$, where $\mathbf{U}$ is an $m \times m$ complex unitary matrix, $\mathbf{\Sigma}$ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and $\mathbf{V}$ is an $n \times n$ complex unitary matrix.

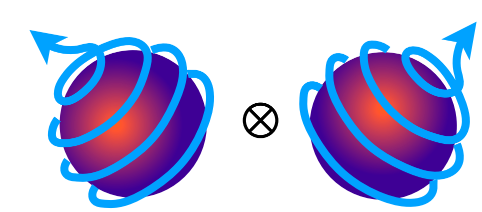
THE REEH-SCHLIEDER THEOREM

V. Discussion

- Finite-dimensional quantum system
- Corollary (Schmidt decomposition): for any state Ψ, we can find an orthonormal basis of $\mathscr{H}_1$ and an orthonormal basis of $\mathscr{H}_2$, s.t.

$$\Psi = \sum_{k=1}^{n} c_k |k\rangle \otimes |k\rangle'$$

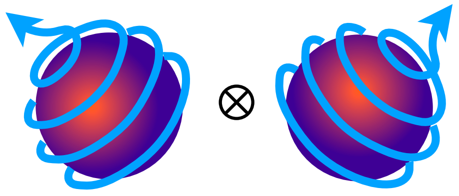
Erhard Schmidt
(1876/01/13-1959/12/06)



THE REEH-SCHLIEDER THEOREM

V. Discussion

- Finite-dimensional quantum system
- When will Ψ be a cyclic and separating vector for $\mathfrak{A}_1$ and $\mathfrak{A}_2$?

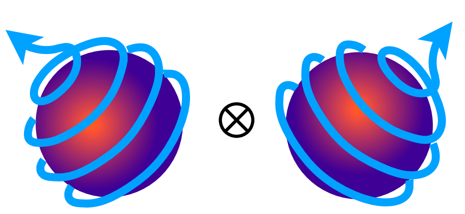


THE REEH-SCHLIEDER THEOREM

V. Discussion

- Finite-dimensional quantum system
- When will Ψ be a cyclic and separating vector for $\mathfrak{A}_1$ and $\mathfrak{A}_2$?

$$\mathfrak{A}_1\Psi \to \left\{ A \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & c_n \end{pmatrix} \middle| A \in \mathbb{C}_{n\times n} \right\}$$



THE REEH-SCHLIEDER THEOREM

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- Finite-dimensional quantum system

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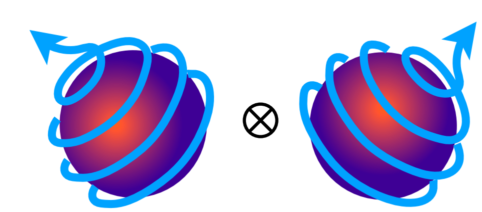
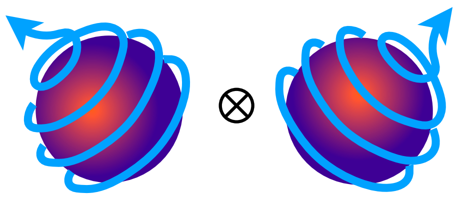
- Ψ is a cyclic (separating) vector for $\mathfrak{A}_1$ ($\mathfrak{A}_2$)
  $\Rightarrow$ rank $C = n$ $(c_1, \cdots, c_n \neq 0)$.



THE REEH-SCHLIEDER THEOREM

V. Discussion

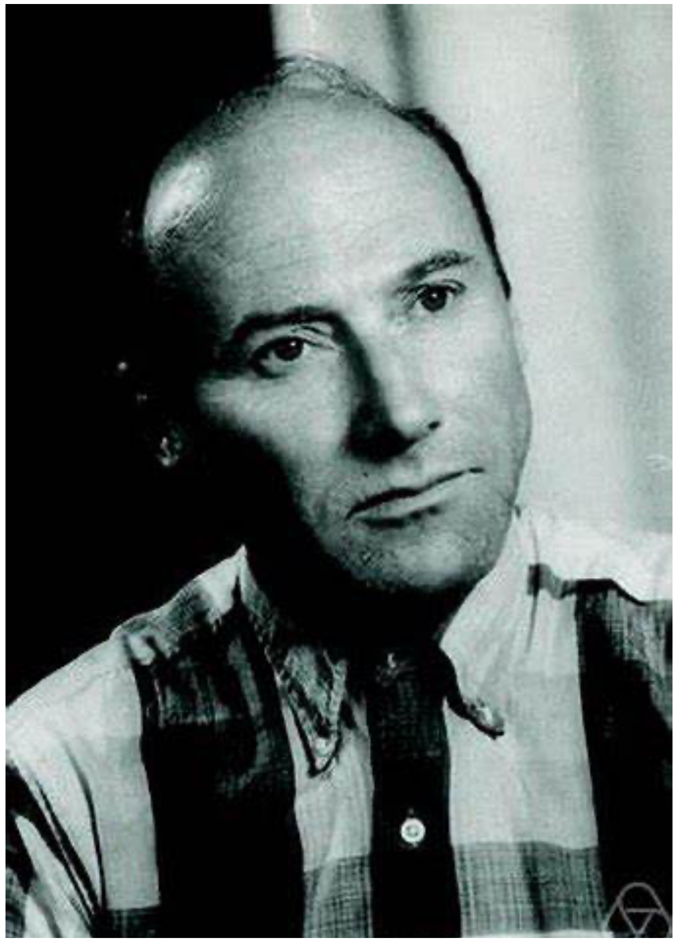
- Finite-dimensional quantum system
- If $\dim \mathscr{H}_1 \neq \dim \mathscr{H}_2$, Ψ can not be a cyclic and separating vector for both $\mathfrak{A}_1$ and $\mathfrak{A}_2$.
- Technically, the Hilbert space of the state vectors of the QFT can be factorized into neither $\mathscr{H}_{\mathscr{U}} \otimes \mathscr{H}_{\mathscr{U}'}$, nor $\bigoplus_\zeta \mathscr{H}^\zeta_{\mathscr{U}} \otimes \mathscr{H}^\zeta_{\mathscr{U}'}$.
- In QFT, the entanglement entropy between adjacent regions has a universal UV divergence, independent of the states considered.



THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- The local algebras (algebraic quantum field theory, AQFT)



Rudolf Haag
(1922/08/17-2016/01/05)

Daniel Kastler
(1926/03/04-2015/07/04)

Huzihiro Araki
荒木 不二洋
(1932/07/28-)

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- The “local algebra” $\mathfrak{A}(\mathscr{U})$ consists of “all operators” supported in $\mathscr{U}$. What does it precisely mean?

    1. Simple operators: polynomials in smeared local fields;

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- The "local algebra" $\mathfrak{A}(\mathscr{U})$ consists of "all operators" supported in $\mathscr{U}$. What does it precisely mean?

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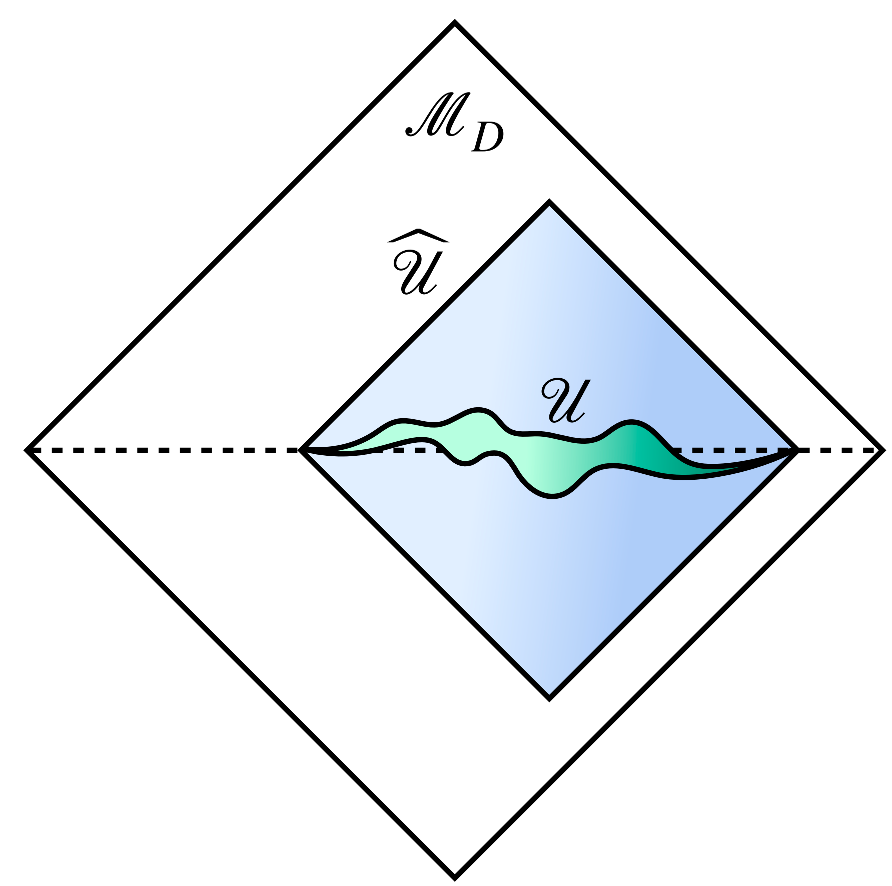
    2. If $\widehat{\mathscr{U}}$ is the domain of dependence of $\mathscr{U}$, then $\mathfrak{A}(\widehat{\mathscr{U}}) = \mathfrak{A}(\mathscr{U})$;



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    2. Bounded operators made from ϕ_f;

    3. And the "limit" points.

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- Why “bounded”?

THE REEH-SCHLIEDER THEOREM

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- Why “bounded”?
- Which “limit”?

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Why “bounded”?
- There is a lot of trouble to define an algebra with unbounded operators.
- An observable measured within finite spacetime region can not touch the “unbounded” part.

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Which “limit”?
- Weak limit versus strong limit:
    - Weak limit: if for $\forall\ |\varphi\rangle, |\psi\rangle \in \mathscr{H}$, $\langle\varphi|\mathbf{a}_n|\psi\rangle \to \langle\varphi|\mathbf{a}|\psi\rangle$, then

$$w - \lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$$

    - Strong limit: if $\forall\ |\psi\rangle \in \mathscr{H}$, $\|\mathbf{a}_n|\psi\rangle - \mathbf{a}|\psi\rangle\| \to 0$, then

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$$

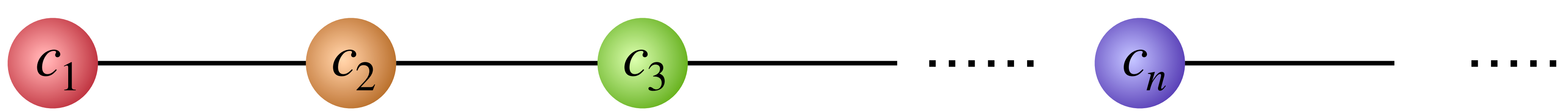
THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- The strong limit condition is stronger than the weak limit condition.

- A strong limit must be a weak limit, and **not** vice versa.

- A counterexample: a sequence $\{\mathbf{a}_n\}$ on $\ell^2(\mathbb{C})$, defined by $\mathbf{a}_n\{c_1, \cdots, c_i, \cdots\} = \{0,\cdots,0,\ c_1, \cdots, c_i, \cdots\}$, then $w - \lim_{n\to\infty} \mathbf{a}_n = \mathbf{0}$. But it is obviously that the strong limit does not exist ($\|\mathbf{a}_n\xi\| = \|\xi\|$).



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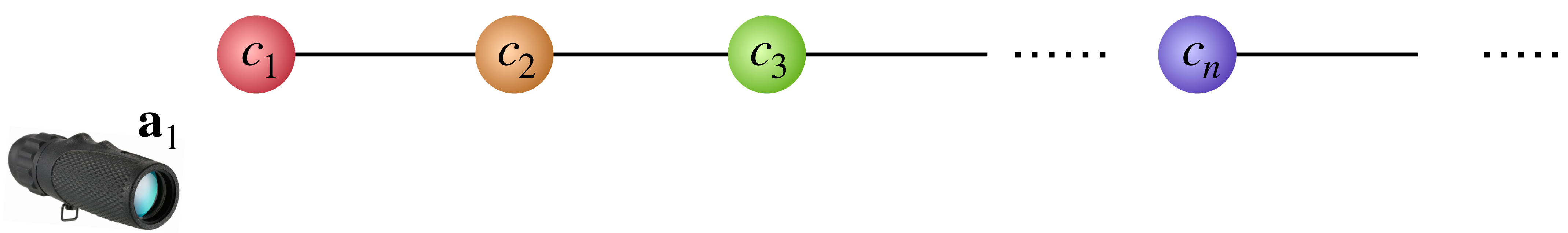

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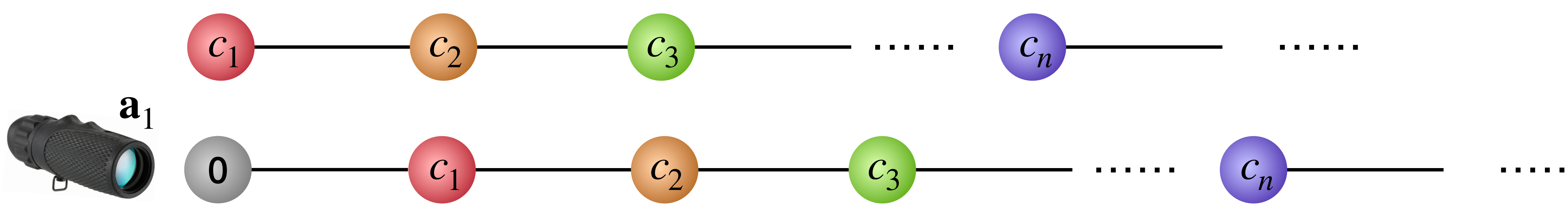

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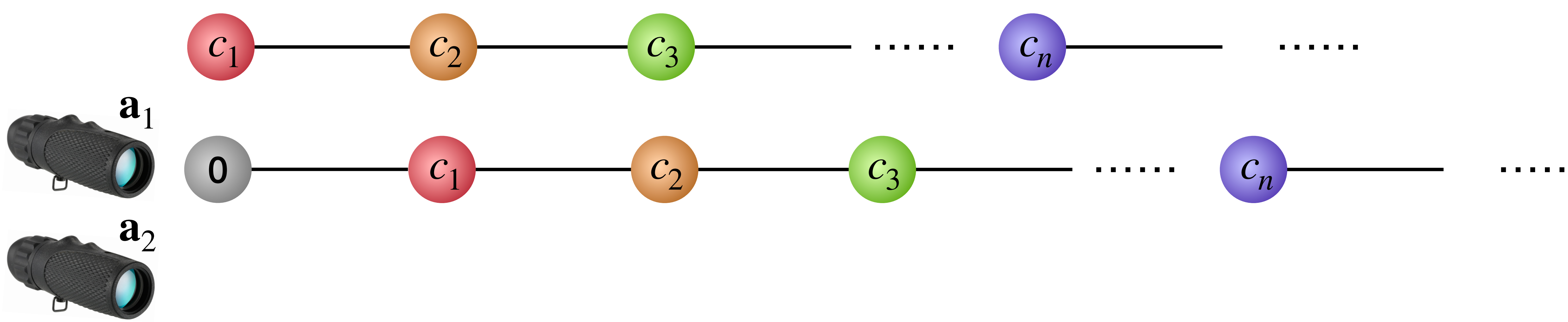

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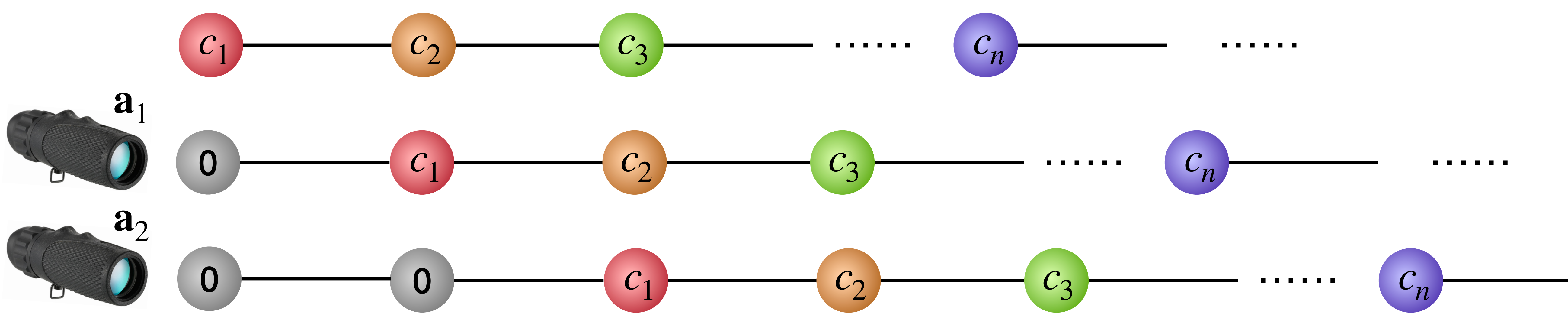

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Many different topologies for operator algebra!
- Norm topology, strong operator topology, strong-*operator topology, σ-strong topology, σ-strong-* topology, weak topology, weak operator topology, weak-*operator topology, σ-weak topology, σ-weak-* topology…

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Physically, the weak limit seems to be more reasonable, since we "measure" an operator $\mathbf{a}$ by measuring the transition amplitude $\langle \varphi \,|\, \mathbf{a} \,|\, \psi \rangle$ induced by $\mathbf{a}$.

- The "local algebra" $\mathfrak{A}(\mathscr{U})$ consists of "all operators" supported in $\mathscr{U}$. What does it precisely mean?

   1. Simple operators: polynomials in smeared local fields;

   2. Bounded operators made from ϕ_f;

   3. And the "limit" points;

   4. Closed under Hermitian conjugate.

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- *-algebra: von Neumann algebra (or W*-algebra, weak limit), C*-algebra (strong limit).

Neumann János Lajos
(1903/12/28-1957/02/08)

Israïl Moyseyovich Gel'fand
(1913/09/02-2009/10/05)

Mark Aronovich Naimark
(1909/12/05-1978/12/30)

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Generic properties of von Neumann algebra
- The commutant of $\mathfrak{A} \in \mathscr{B}(\mathscr{H})$

$$\mathfrak{A}' \equiv \{\mathbf{a}' \in \mathscr{B}(\mathscr{H}) \mid \forall\ \mathbf{a} \in \mathfrak{A},\ [\mathbf{a}', \mathbf{a}] = \mathbf{0}\}$$

$$\mathfrak{A} \subseteq \mathfrak{A}'' \equiv (\mathfrak{A}')'$$

- The commutant $\mathfrak{A}'$ of any *-algebra $\mathfrak{A}$ is a von Neumann algebra

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$$\therefore\ \langle\varphi\,|\,[\mathbf{a}', \mathbf{a}]\,|\,\psi\rangle = \langle\varphi\,|\,\mathbf{a}'\mathbf{a}\,|\,\psi\rangle - \langle\varphi\,|\,\mathbf{a}\mathbf{a}'\,|\,\psi\rangle = \lim_{n\to\infty} \big(\langle\varphi\,|\,\mathbf{a}'_n\mathbf{a}\,|\,\psi\rangle - \langle\varphi\,|\,\mathbf{a}\mathbf{a}'_n\,|\,\psi\rangle\big) = 0$$

THE REEH-SCHLIEDER THEOREM

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$$\therefore\ [\mathbf{a}', \mathbf{a}] = 0$$

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Generic properties of von Neumann algebra
- The commutant of $\mathfrak{A} \in \mathscr{B}(\mathscr{H})$

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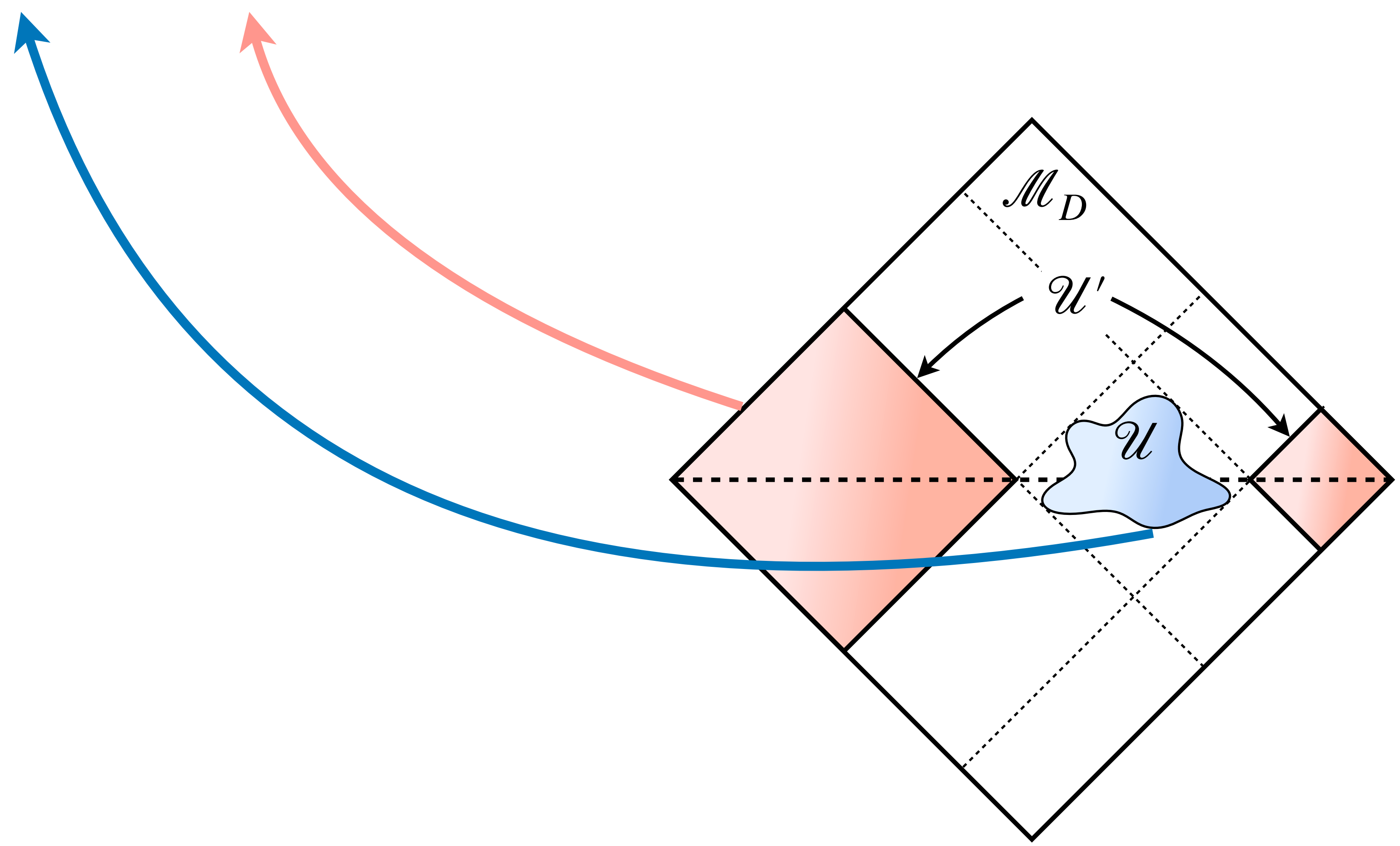
- The commutant $\mathfrak{A}'$ of any *-algebra $\mathfrak{A}$ is a von Neumann algebra
- von Neumann's theorem: if $\mathfrak{A}$ is a von Neumann algebra, then $\mathfrak{A} = \mathfrak{A}''$

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Causality of the local algebra

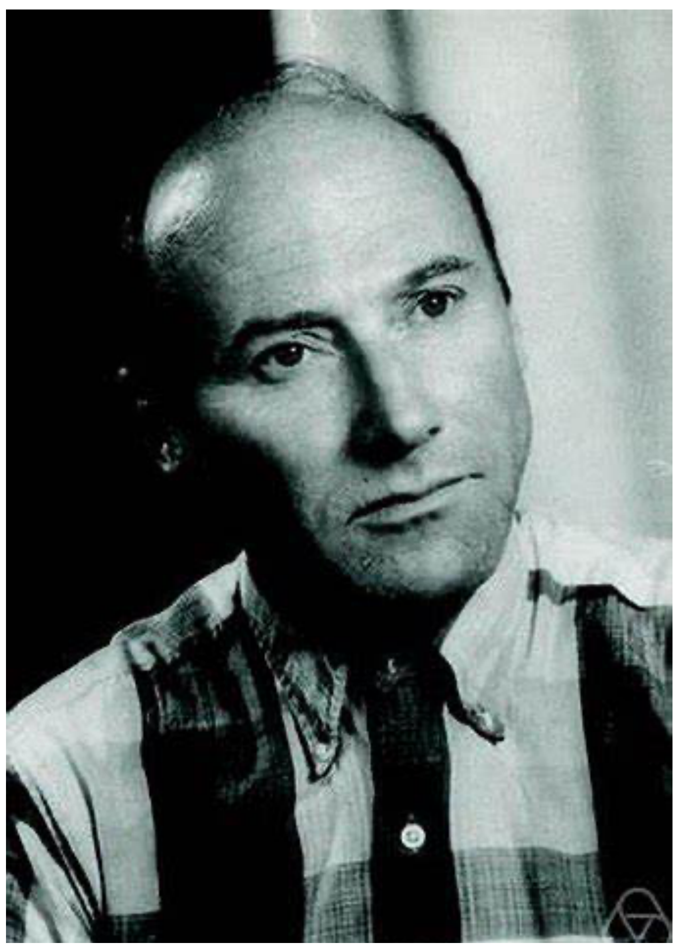
$$[\mathfrak{A}(\mathscr{U}), \mathfrak{A}(\mathscr{U}')] = 0, \quad \mathfrak{A}(\mathscr{U}') \subseteq \mathfrak{A}(\mathscr{U})'$$



THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Haag duality (Haag & Schroer (1962))



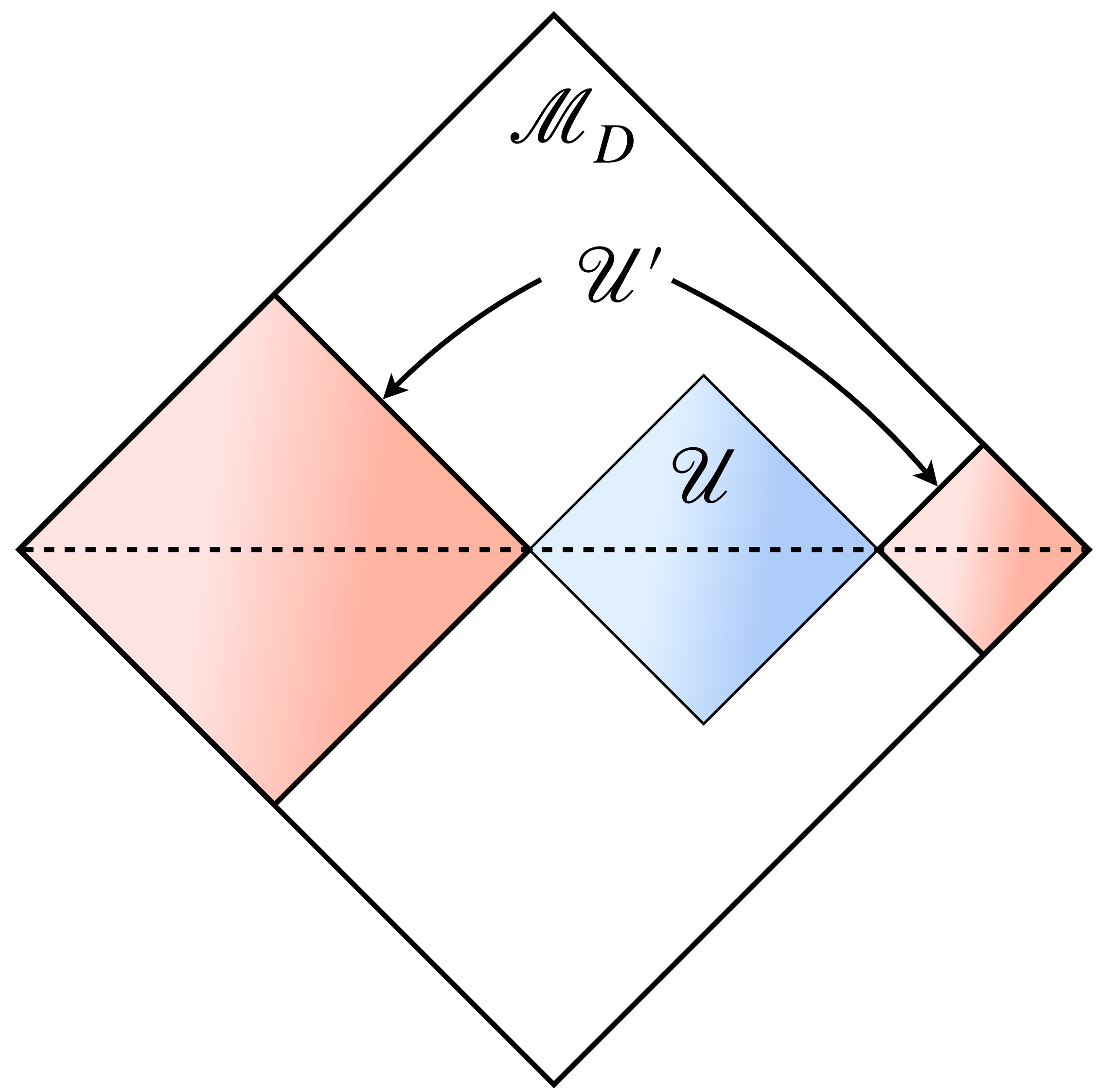
Rudolf Haag
(1922/08/17-2016/01/05)

Bert Schroer
(1933/11/10-)

THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Haag duality (Haag & Schroer (1962)): if $\mathscr{U}$ and $\mathscr{U}'$ are causal complements, then $\mathfrak{A}(\mathscr{U})' = \mathfrak{A}(\mathscr{U}')$.



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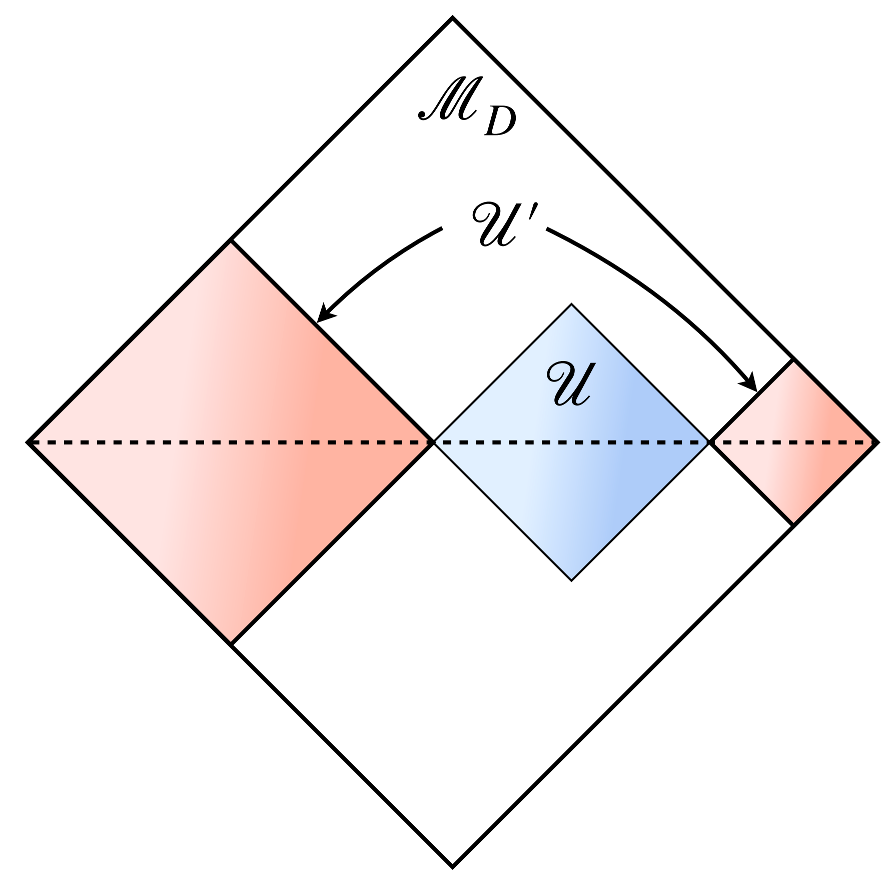
- Haag duality (Haag & Schroer (1962)): if $\mathscr{U}$ and $\mathscr{U}'$ are causal complements, then $\mathfrak{A}(\mathscr{U})' = \mathfrak{A}(\mathscr{U}')$.

- If $\mathscr{U}$ is the union of $\mathscr{U}_\alpha$, then $\mathfrak{A}(\mathscr{U})$ is the smallest von Neumann algebra containing all $\mathfrak{A}(\mathscr{U}_\alpha)$.

- If $\mathscr{U}_1$ and $\mathscr{U}_2$ are two open sets, then $\mathfrak{A}(\mathscr{U}_1 \cap \mathscr{U}_2) = \mathfrak{A}(\mathscr{U}_1) \cap \mathfrak{A}(\mathscr{U}_2)$.



THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

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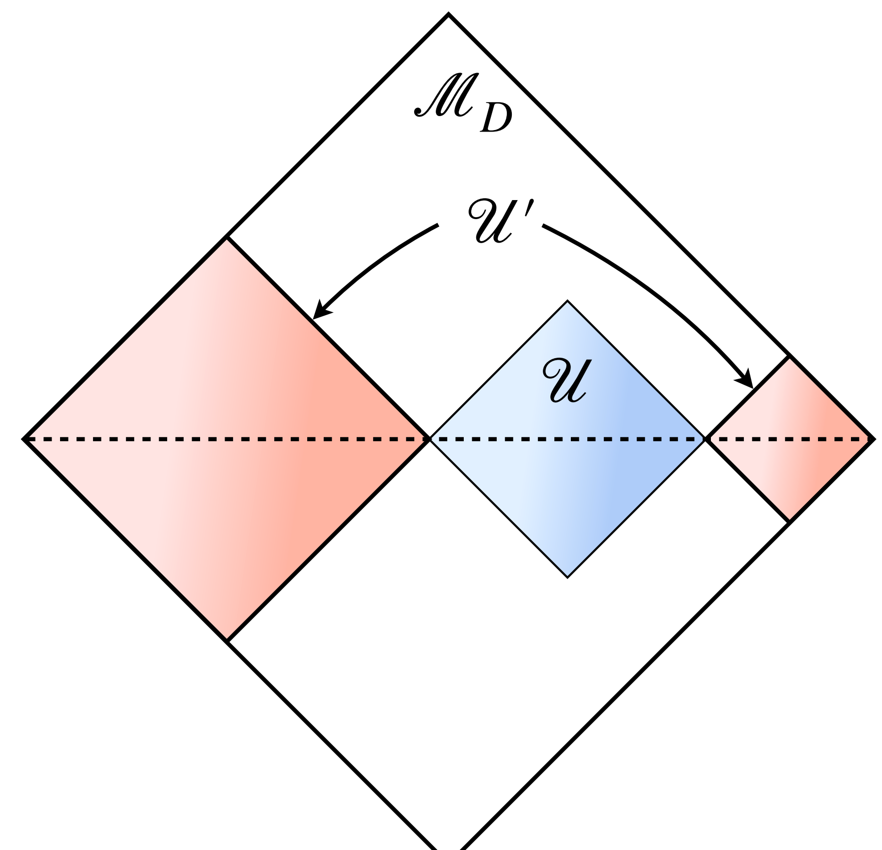
- There are some theories, in which the Haag duality and the two postulates fail.



THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- **Important conclusion**: if $\mathfrak{A}$ and $\mathfrak{A}'$ are commutant, then a vector $|\Omega\rangle \in \mathscr{H}$ is separating for $\mathfrak{A}$ iff it is cyclic for $\mathfrak{A}'$, and vice versa.

- "$\Rightarrow$": we have proved it.

- "$\Leftarrow$" (contradiction): if $|\Omega\rangle$ is not cyclic vector for $\mathfrak{A}'$, then

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$$\exists\ |\psi\rangle \in \mathscr{H},\ |\psi\rangle \neq \mathbf{a}'|\Omega\rangle\ \ (\mathbf{a}' \in \mathfrak{A}')$$

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$$\exists\ |\psi\rangle \in \mathscr{H},\ |\psi\rangle \neq \mathbf{a}'|\Omega\rangle\ (\mathbf{a}' \in \mathfrak{A}')$$

  So there is orthogonal decomposition $\mathscr{H} = \mathscr{H}^{\perp} \oplus \overline{\mathfrak{A}'|\Omega\rangle}$, and the (bounded) projective operator $\Pi : \mathscr{H} \to \mathscr{H}^{\perp}$, $\Pi^2 = \Pi \neq \mathbf{0}$.

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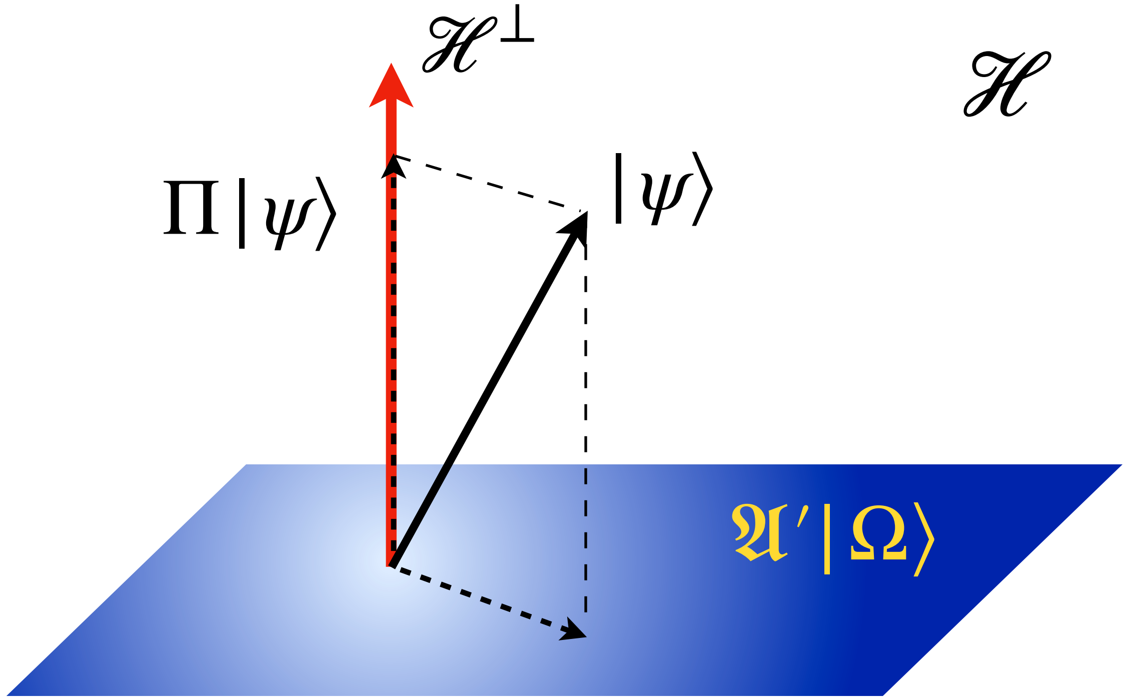
$$\begin{aligned} |\,\psi\rangle &= \Pi\,|\,\psi\rangle + (\mathbf{1} - \Pi)\,|\,\psi\rangle \\ &= \Pi\,|\,\psi\rangle + \mathbf{a}'_{\psi}\,|\,\Omega\rangle \end{aligned}$$



THE REEH-SCHLIEDER THEOREM

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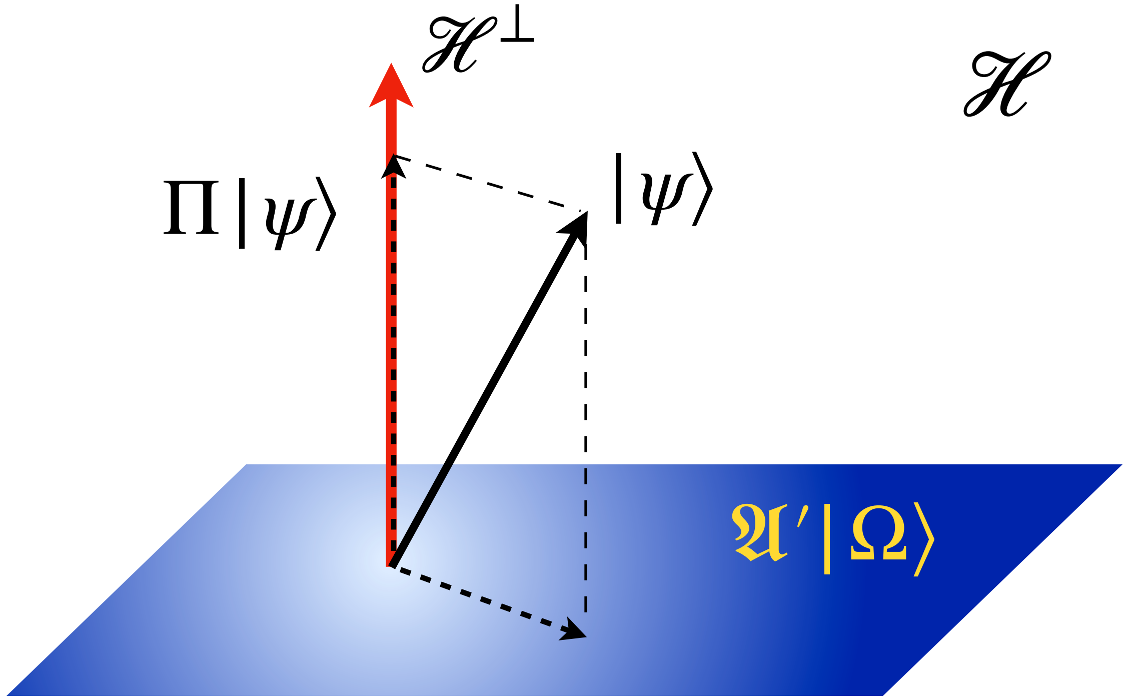
$$
\begin{aligned}
|\psi\rangle &= \Pi|\psi\rangle + (\mathbf{1} - \Pi)|\psi\rangle \\
&= \Pi|\psi\rangle + \mathbf{a}'_{\psi}|\Omega\rangle
\end{aligned}
$$

$$
\begin{aligned}
\Pi\mathbf{a}'|\psi\rangle &= \Pi\mathbf{a}'\Pi|\psi\rangle + \Pi\mathbf{a}'\mathbf{a}'_{\psi}|\Omega\rangle \\
&= \Pi\mathbf{a}'\Pi|\psi\rangle
\end{aligned}
$$



THE REEH-SCHLIEDER THEOREM

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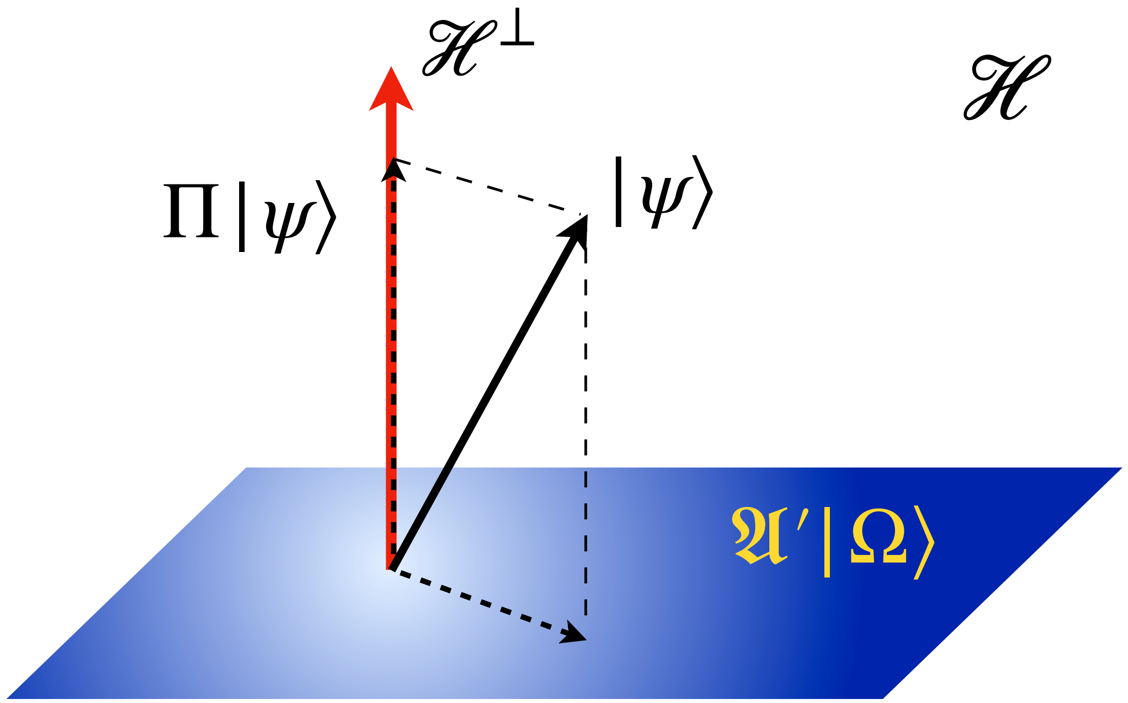
$$\begin{aligned}\Pi\mathbf{a}'|\psi\rangle &= \Pi\mathbf{a}'\Pi|\psi\rangle + \Pi\mathbf{a}'\mathbf{a}'_\psi|\Omega\rangle \\ &= \Pi\mathbf{a}'\Pi|\psi\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{a}'\Pi|\psi\rangle &= \mathbf{a}'\Pi|\psi\rangle + \mathbf{a}'\Pi\mathbf{a}'_\psi|\Omega\rangle \\ &= \mathbf{a}'\Pi|\psi\rangle\end{aligned}$$



THE REEH-SCHLIEDER THEOREM

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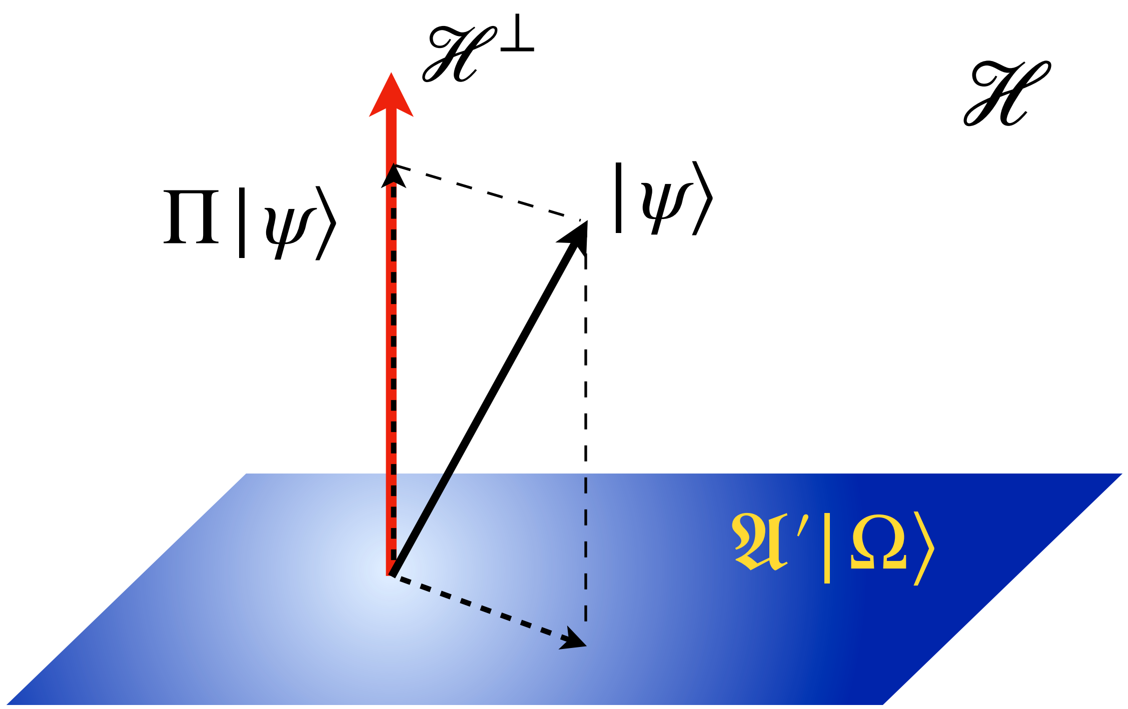

If $\mathbf{a}'\Pi|\psi\rangle$ has nonzero component in $\mathfrak{A}'|\Omega\rangle$, then $\exists\ \mathbf{b}' \in \mathfrak{A}'$ satisfies $\langle\Omega|\mathbf{b}'^{\dagger}\mathbf{a}'\Pi|\psi\rangle \neq 0$, so $\Pi|\psi\rangle$ has nonzero component $\mathbf{a}'^{\dagger}\mathbf{b}'|\Omega\rangle$ in $\mathfrak{A}'|\Omega\rangle$, which is impossible.

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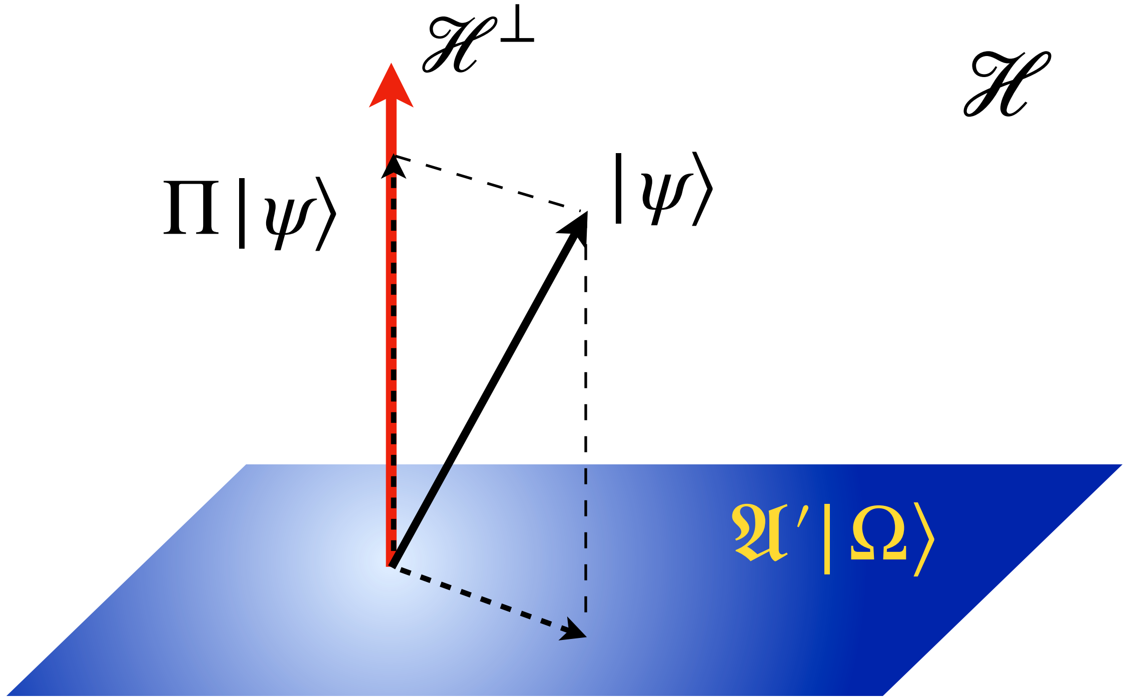

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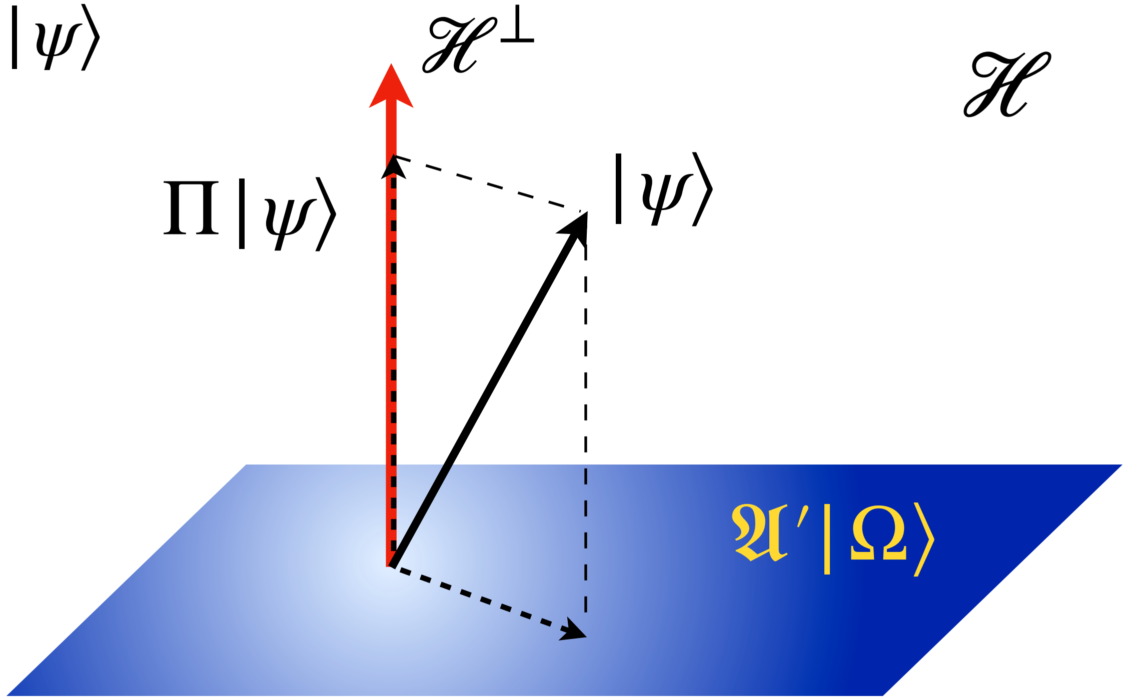
$$\therefore\ \mathbf{a}'\Pi|\psi\rangle = \mathbf{a}'\Pi|\psi\rangle = \Pi\mathbf{a}'\Pi|\psi\rangle = \Pi\mathbf{a}'|\psi\rangle$$

$\Rightarrow$ for $\forall\ \mathbf{a}' \in \mathfrak{A}'$, $[\mathbf{a}', \Pi] = \mathbf{0}$



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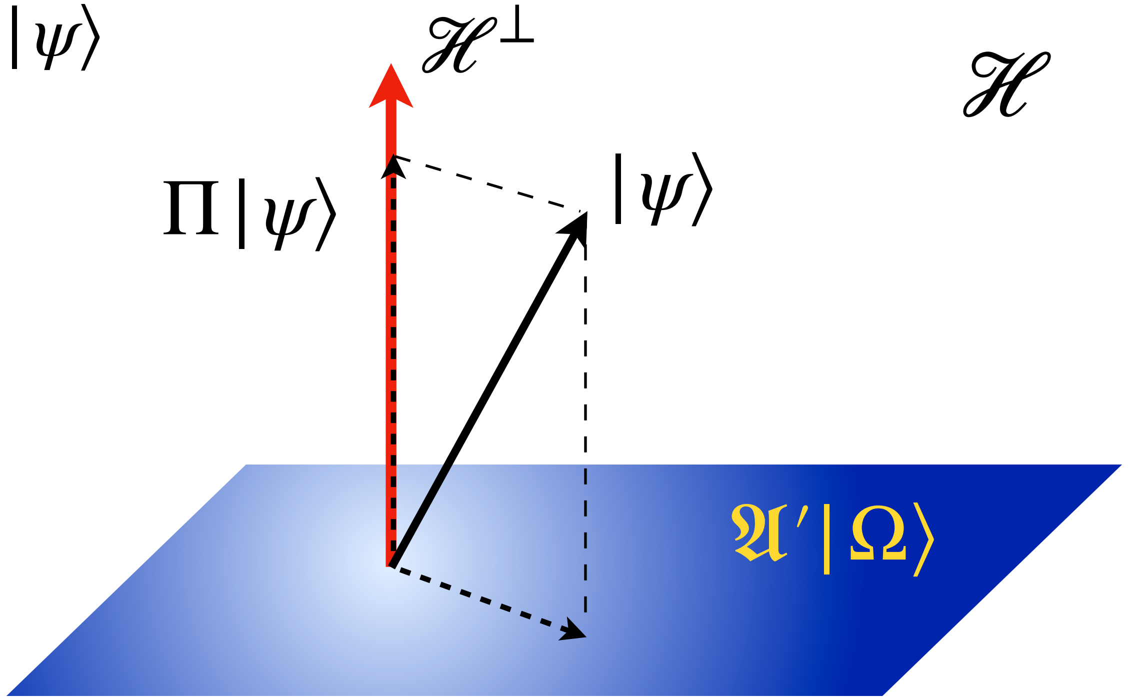
$$\Rightarrow \mathbf{0} \neq \Pi \in (\mathfrak{A}')' = \mathfrak{A}$$



THE REEH-SCHLIEDER THEOREM

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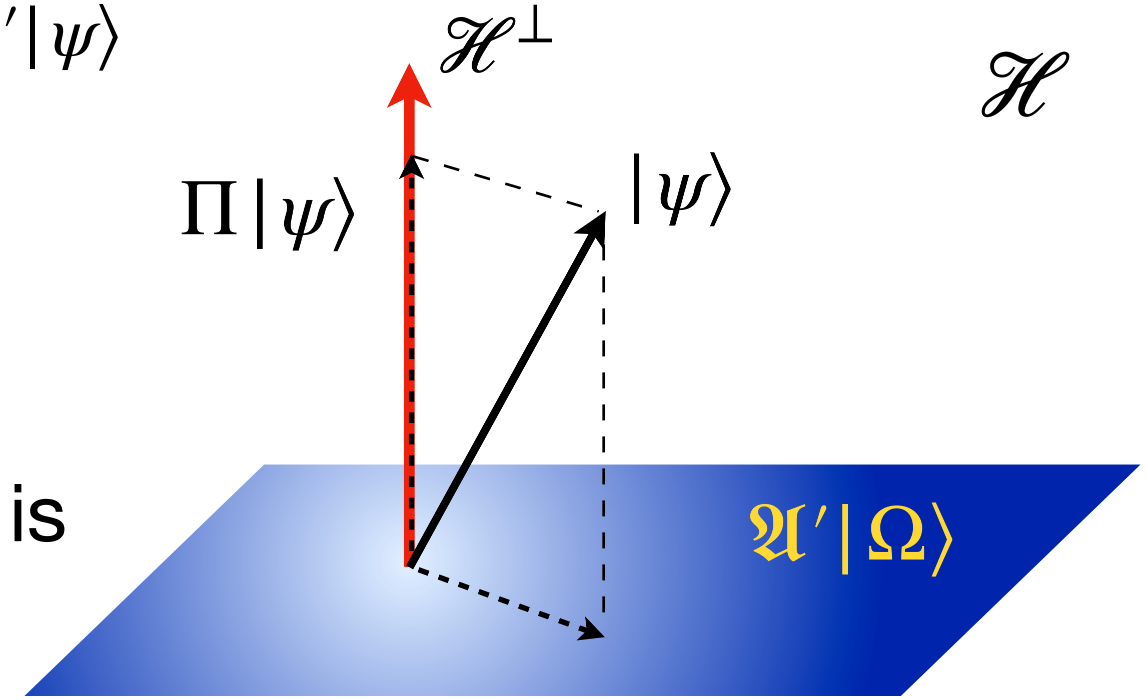
- **Important conclusion**: if $\mathfrak{A}$ and $\mathfrak{A}'$ are commutant, then a vector $|\,\Omega\rangle \in \mathscr{H}$ is separating for $\mathfrak{A}$ iff it is cyclic for $\mathfrak{A}'$, and vice versa.

$$\therefore\ \ \mathbf{a}'\Pi\,|\,\psi\rangle = \mathbf{a}'\Pi\,|\,\psi\rangle = \Pi\mathbf{a}'\Pi\,|\,\psi\rangle = \Pi\mathbf{a}'\,|\,\psi\rangle$$

$\Rightarrow$ for $\forall\ \mathbf{a}' \in \mathfrak{A}'$, $[\mathbf{a}', \Pi] = \mathbf{0}$

$$\Rightarrow \mathbf{0} \neq \Pi \in (\mathfrak{A}')' = \mathfrak{A}$$

So $|\,\Omega\rangle$ is not cyclic for $\mathfrak{A}'$ $\Rightarrow$ it is not separating for $\mathfrak{A}$.

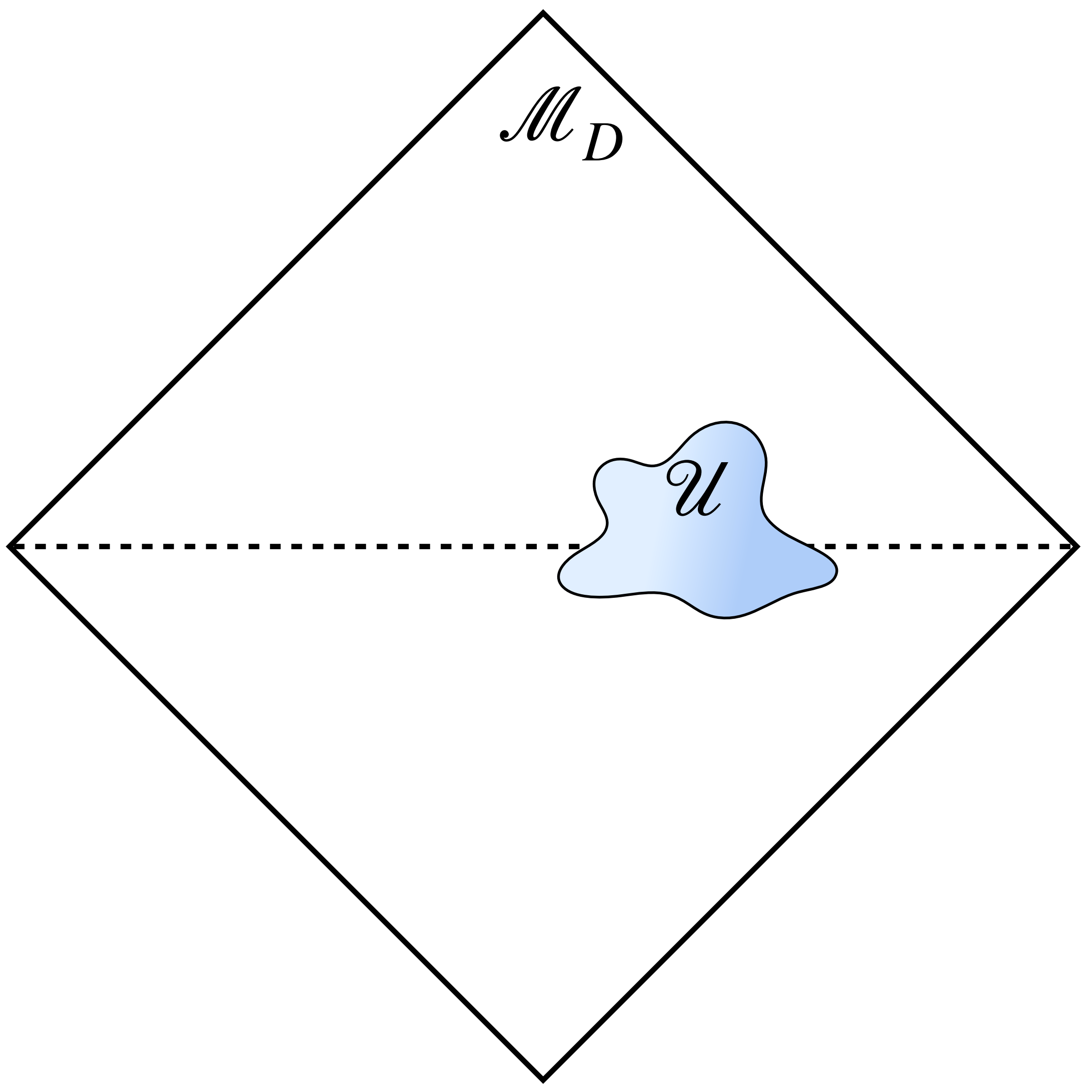


THE REEH-SCHLIEDER THEOREM

V. The Local Algebra

- Conclusion

Local Algebra



Spacetime

THE REEH-SCHLIEDER THEOREM

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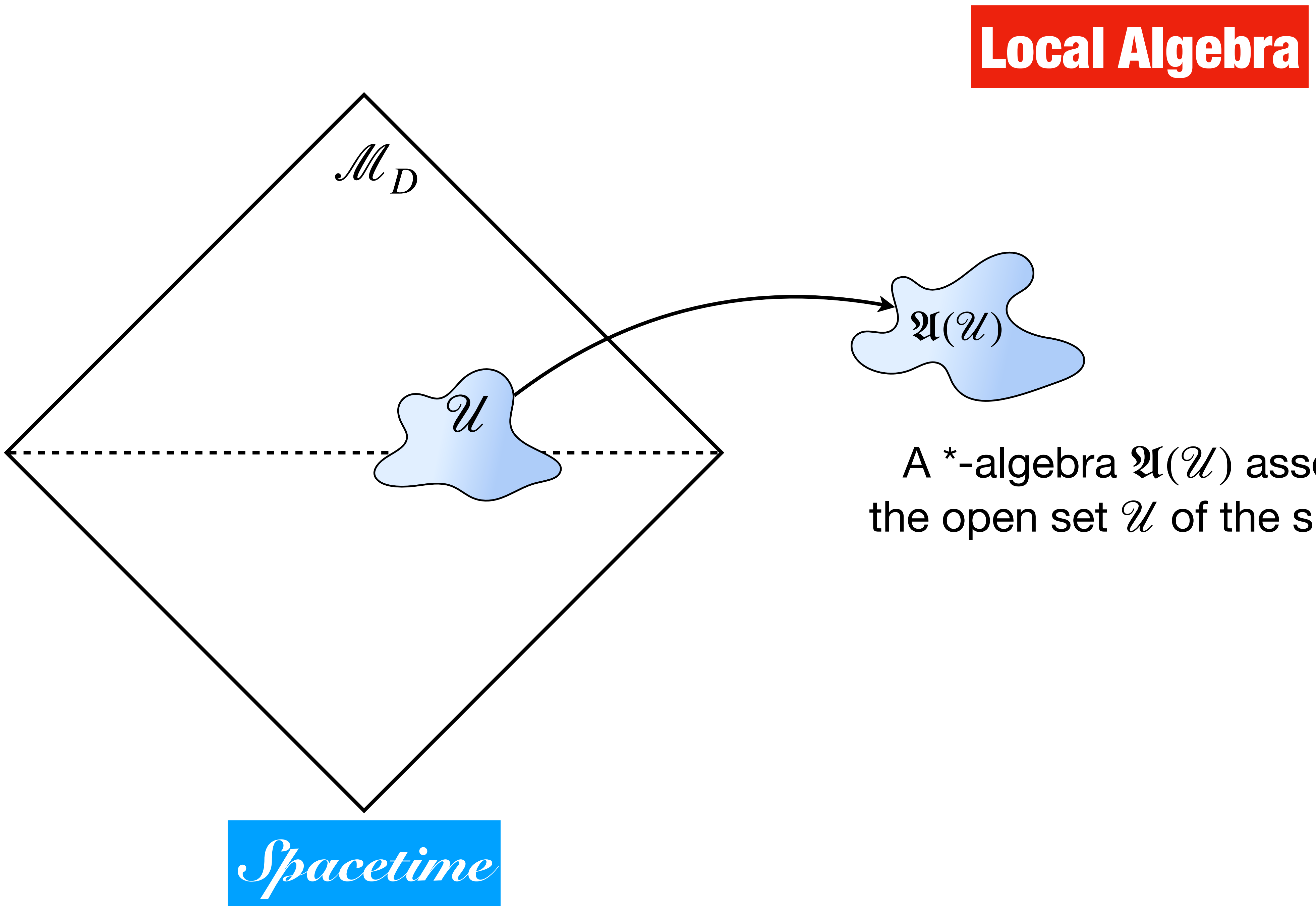
- Conclusion



A *-algebra $\mathfrak{A}(\mathscr{U})$ associates with the open set $\mathscr{U}$ of the spacetime $\mathscr{M}_D$

THE REEH-SCHLIEDER THEOREM

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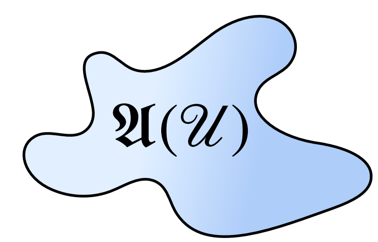
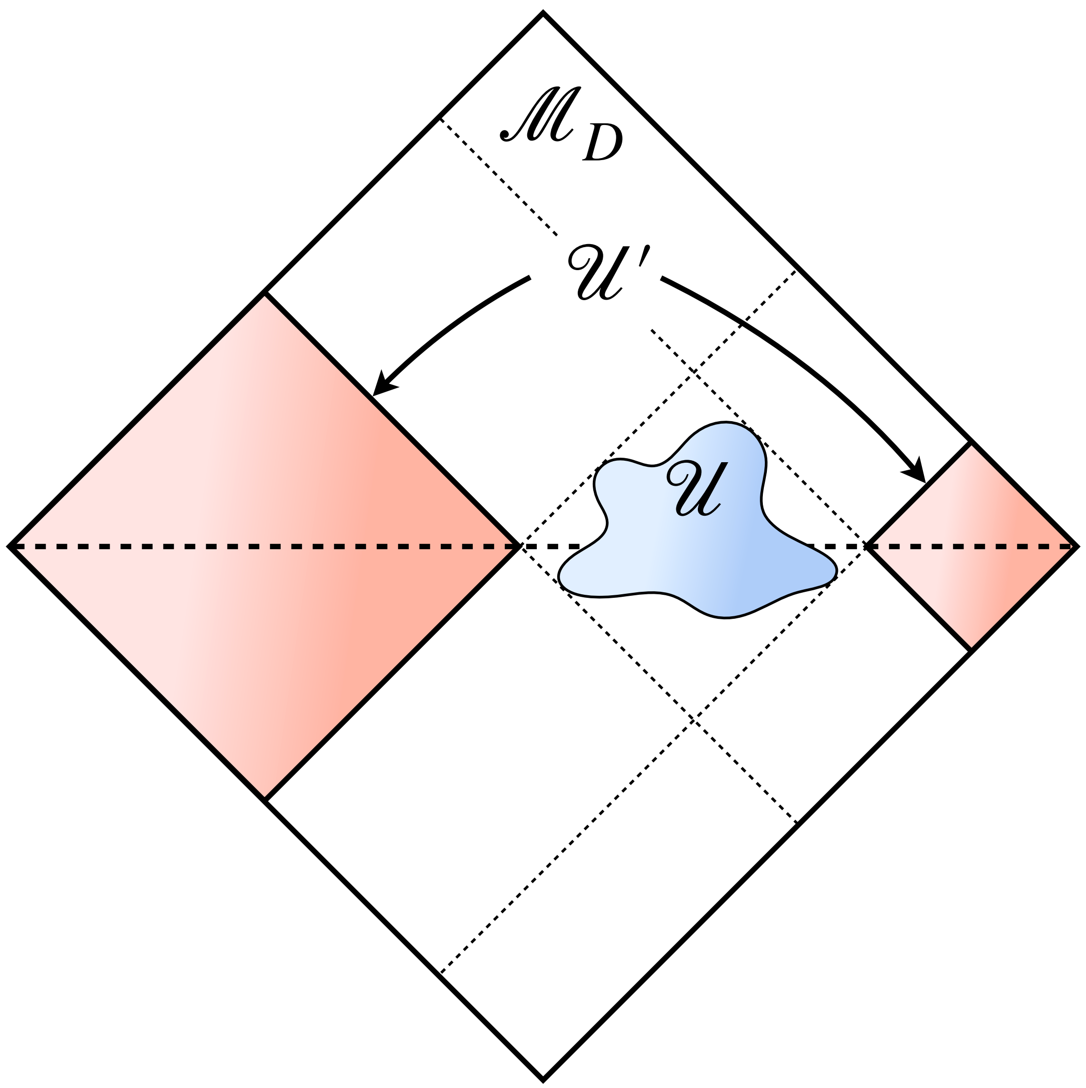
Local Algebra





Spacetime

THE REEH-SCHLIEDER THEOREM

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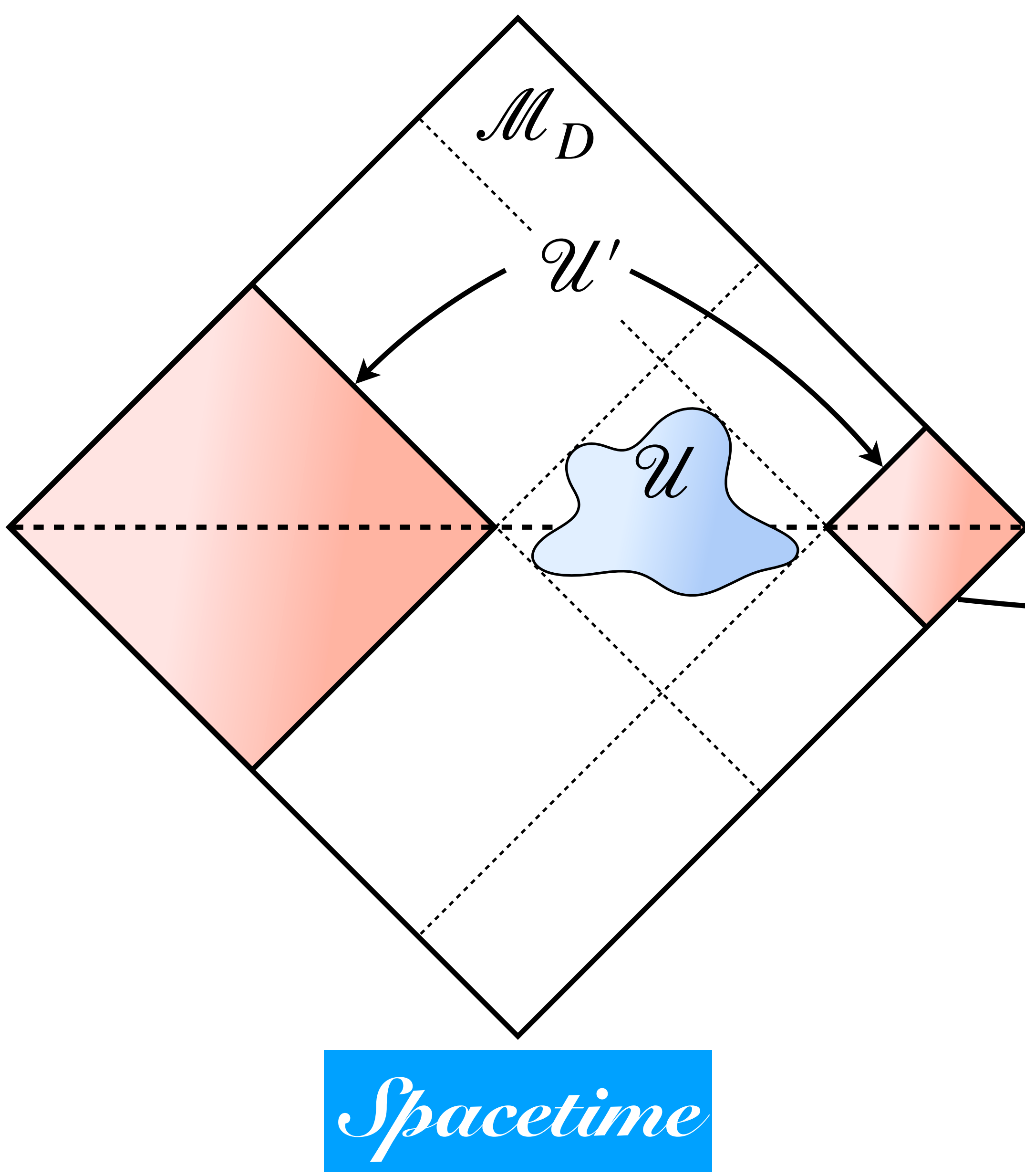
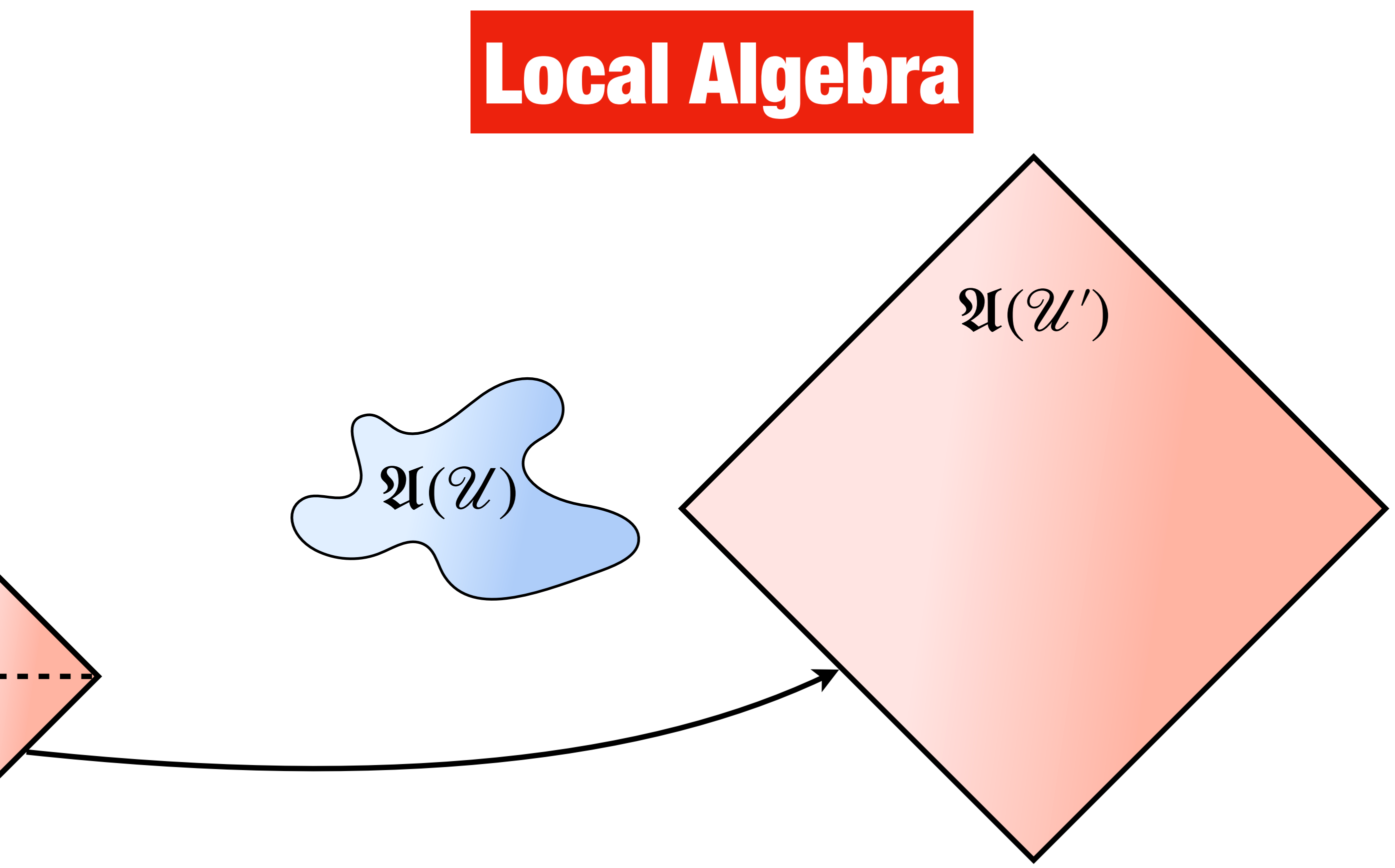
- Conclusion





A von Neumann algebra $\mathfrak{A}(\mathscr{U}')$ associates with the causal complement $\mathscr{U}'$ of $\mathscr{U}$

Causality: $\mathfrak{A}(\mathscr{U}') \subseteq \mathfrak{A}(\mathscr{U})'$

THE REEH-SCHLIEDER THEOREM

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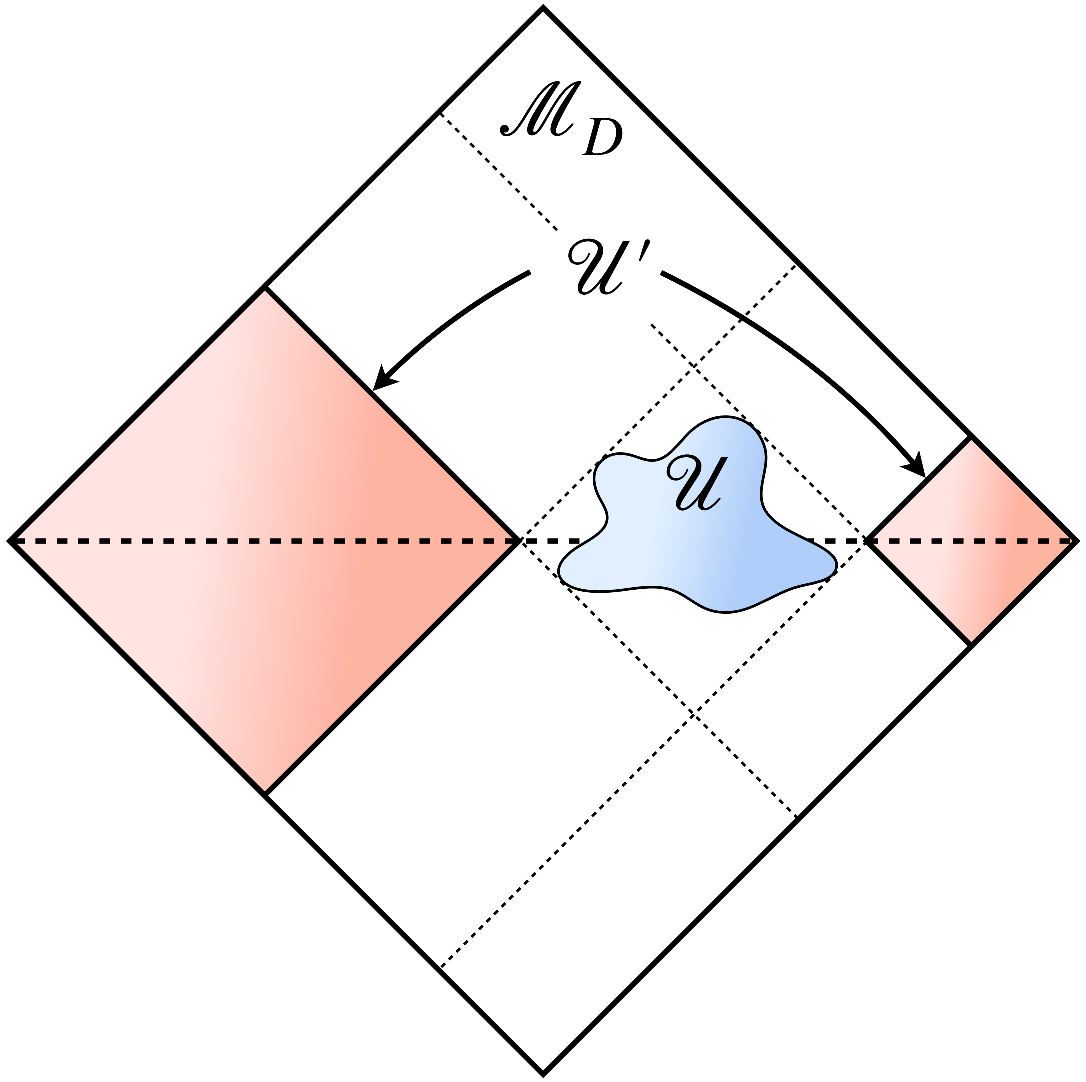
Local Algebra



Spacetime

THE REEH-SCHLIEDER THEOREM

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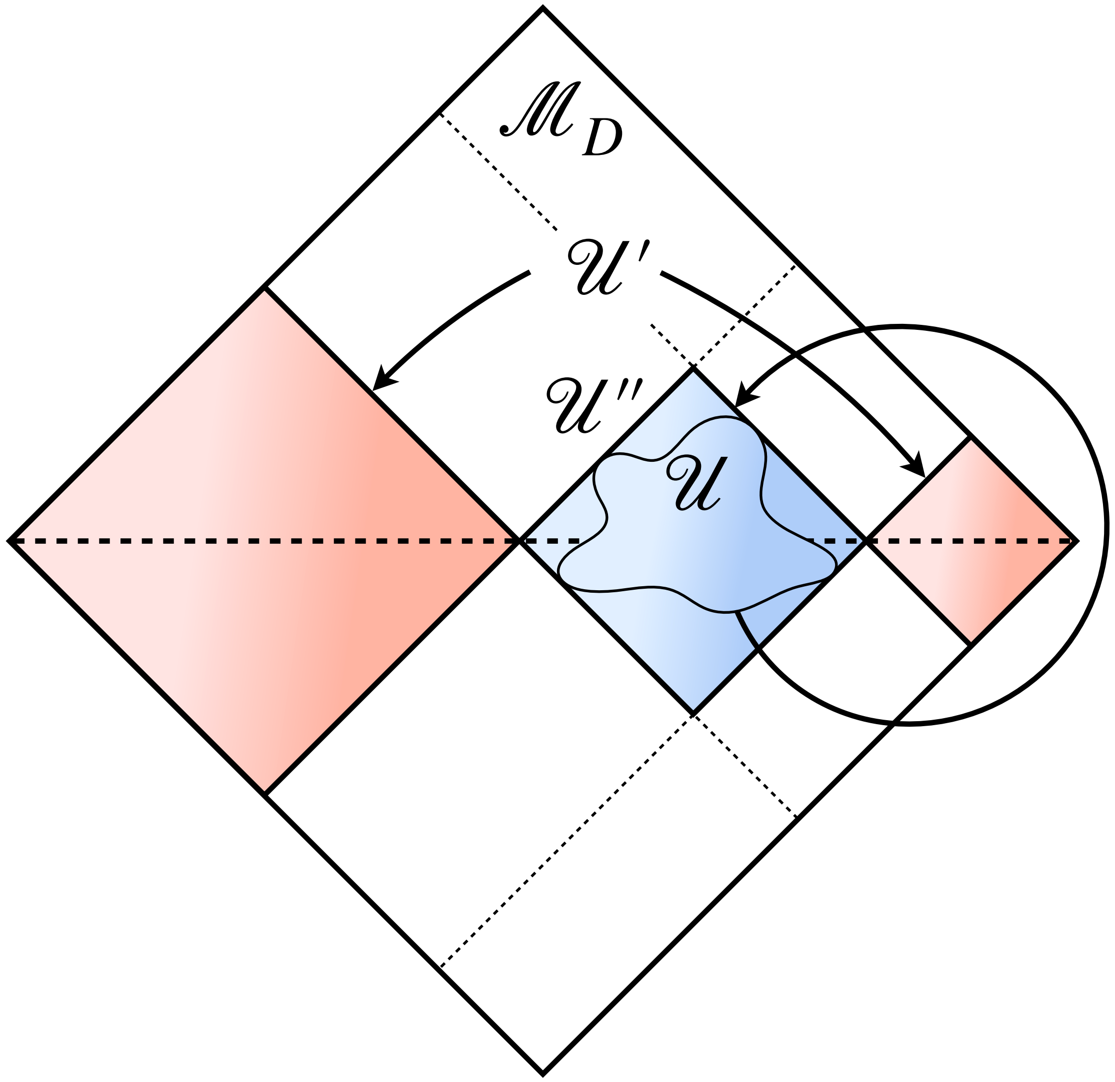

Spacetime

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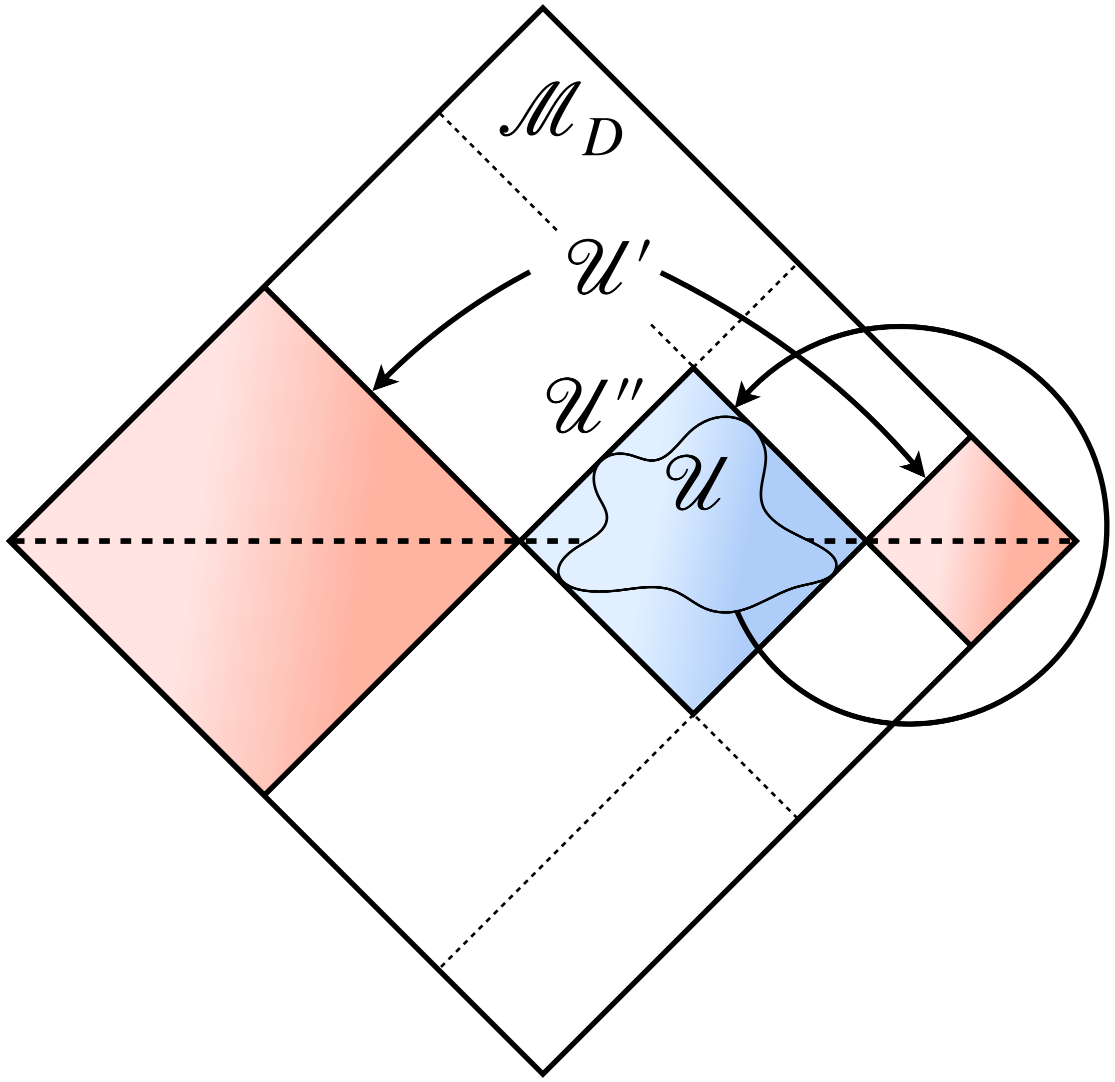

Spacetime

$\mathscr{U}''$ always contains $\mathscr{U}$, $\mathscr{U}''' = \mathscr{U}'$

THE REEH-SCHLIEDER THEOREM

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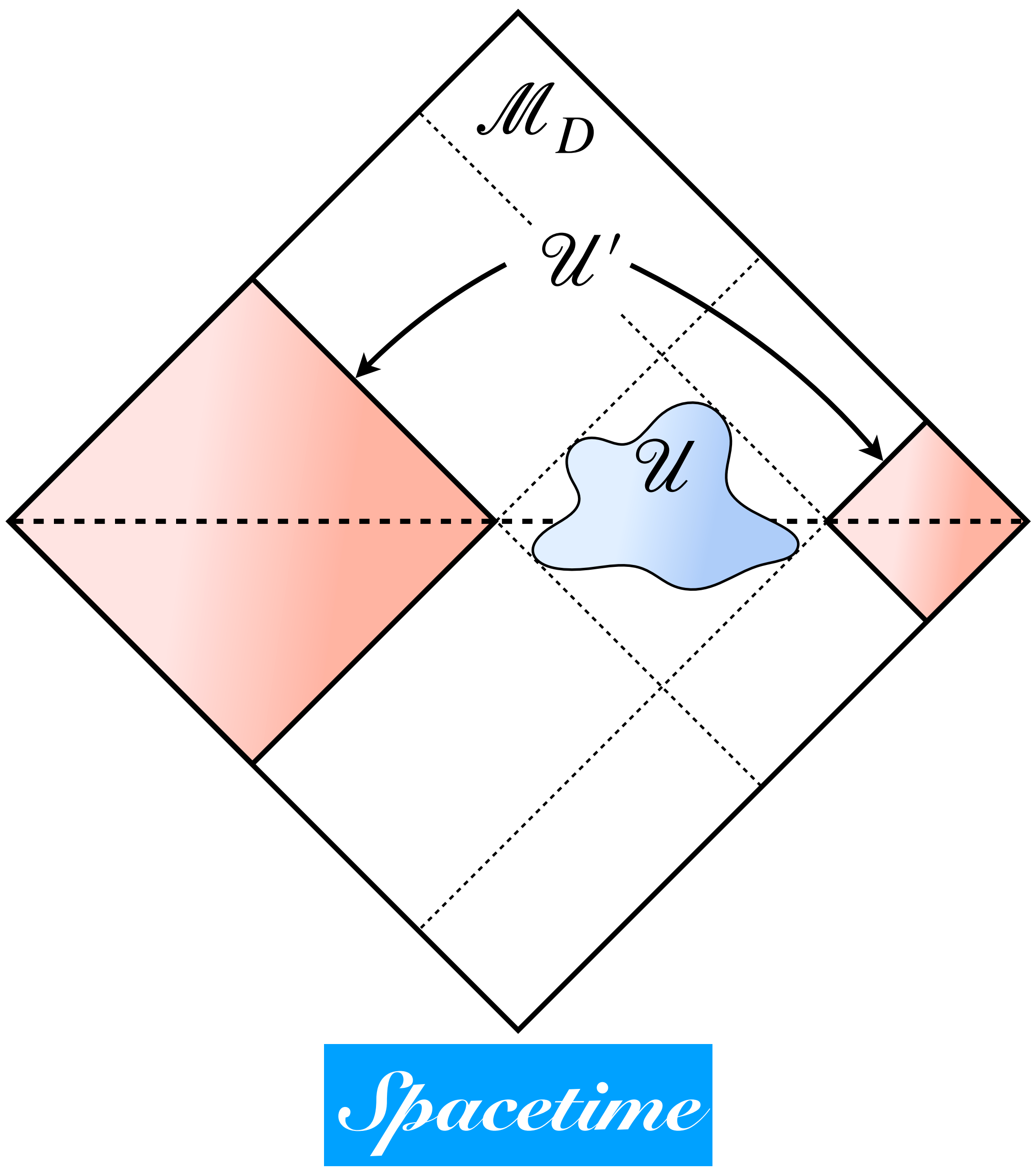

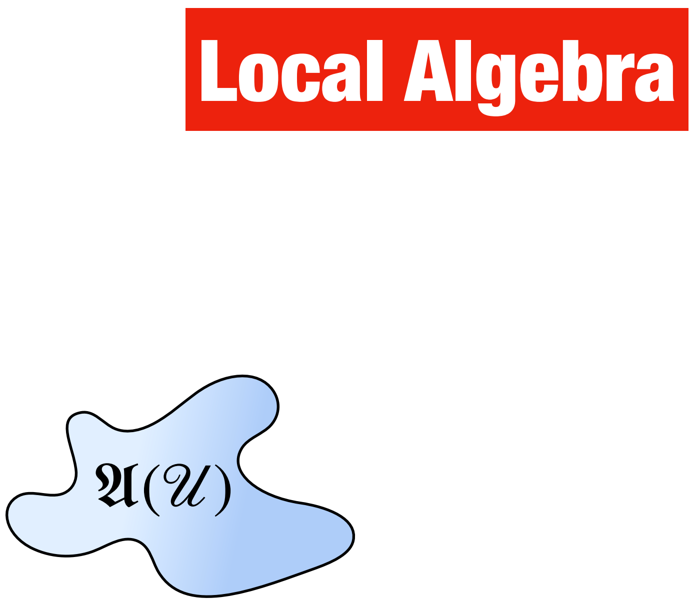

THE REEH-SCHLIEDER THEOREM

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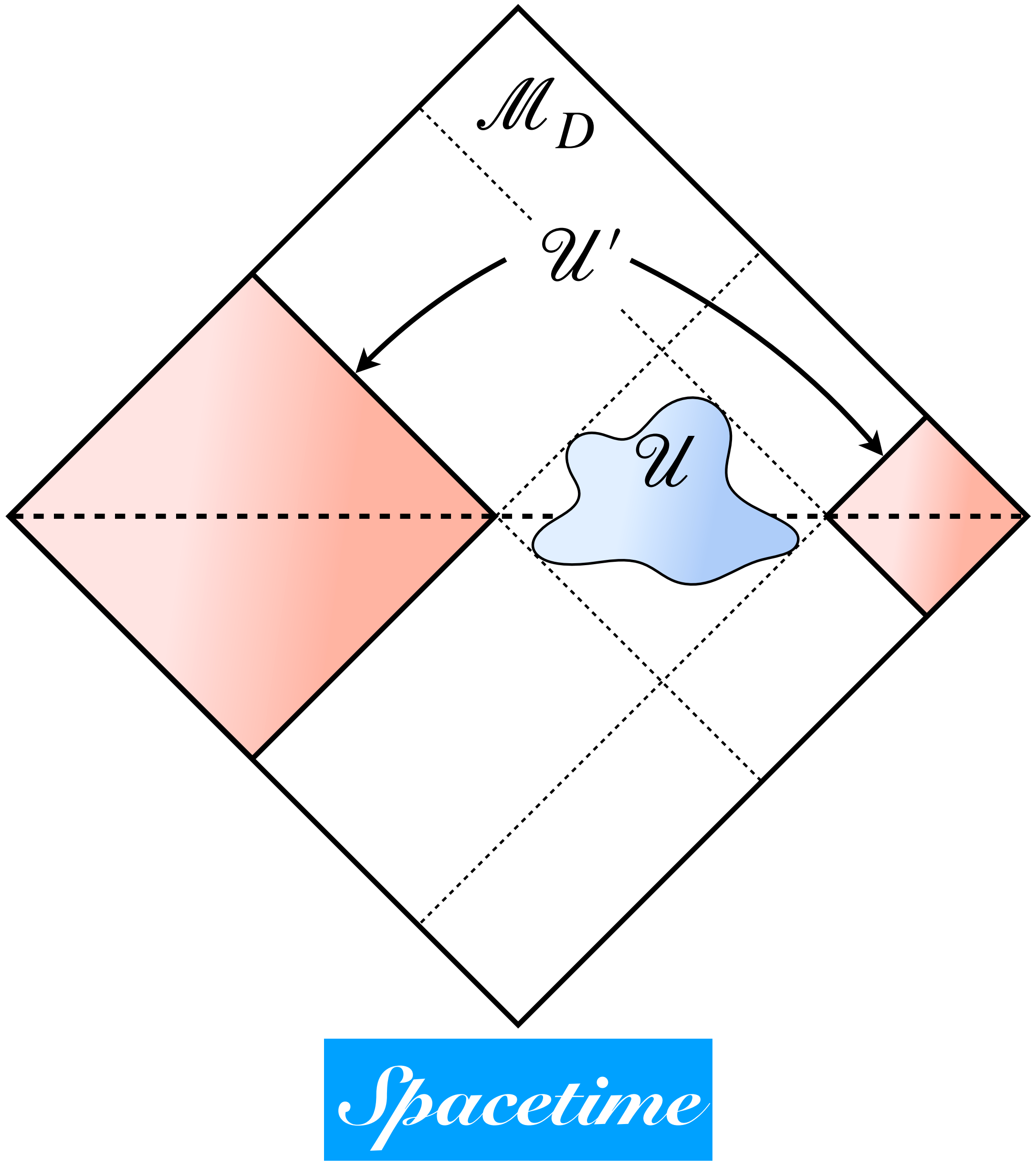

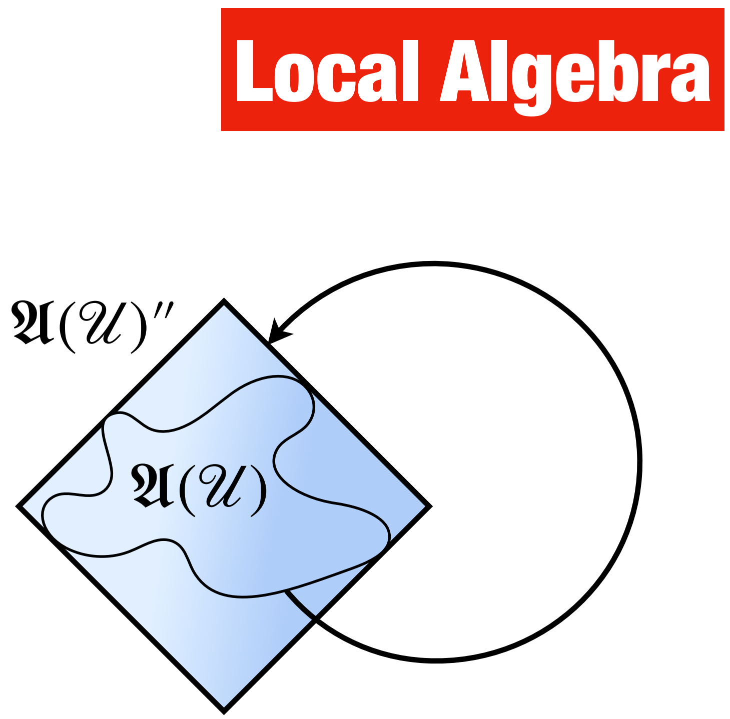

THE REEH-SCHLIEDER THEOREM

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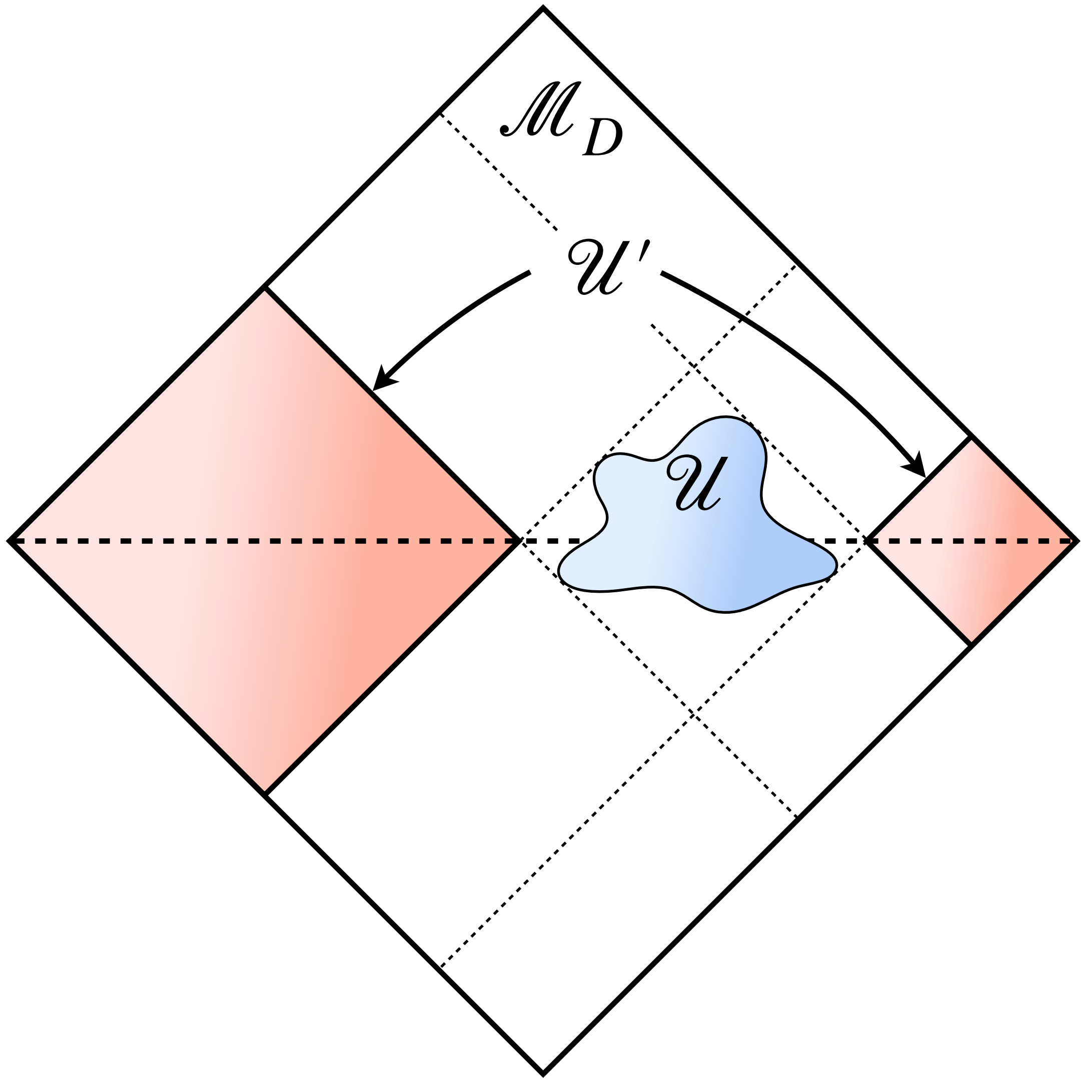
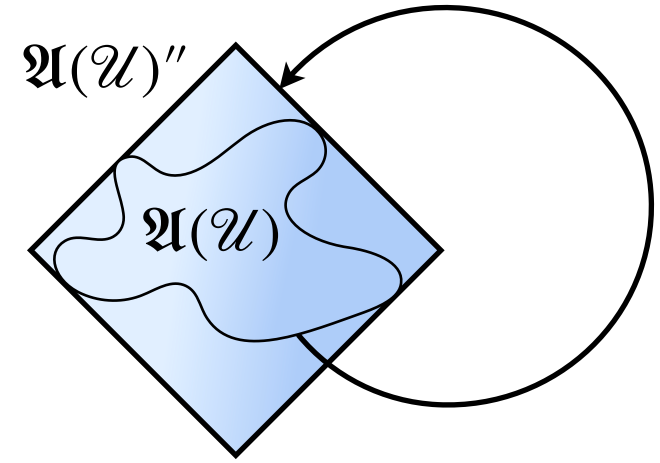


Spacetime



$\mathfrak{A}(\mathscr{U})''$ is the smallest von Neumann algebra contains $\mathfrak{A}(\mathscr{U})$

THE REEH-SCHLIEDER THEOREM

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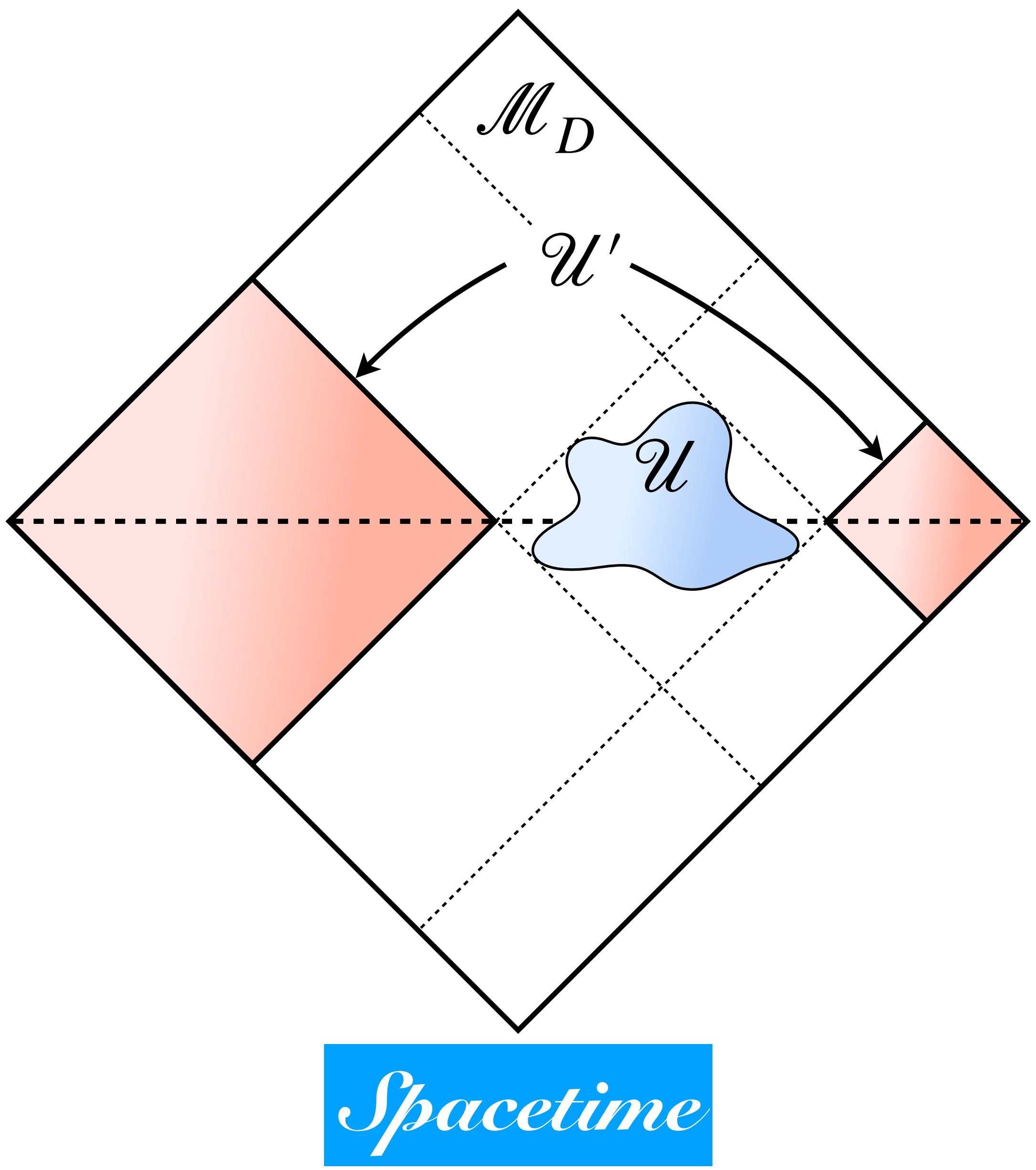
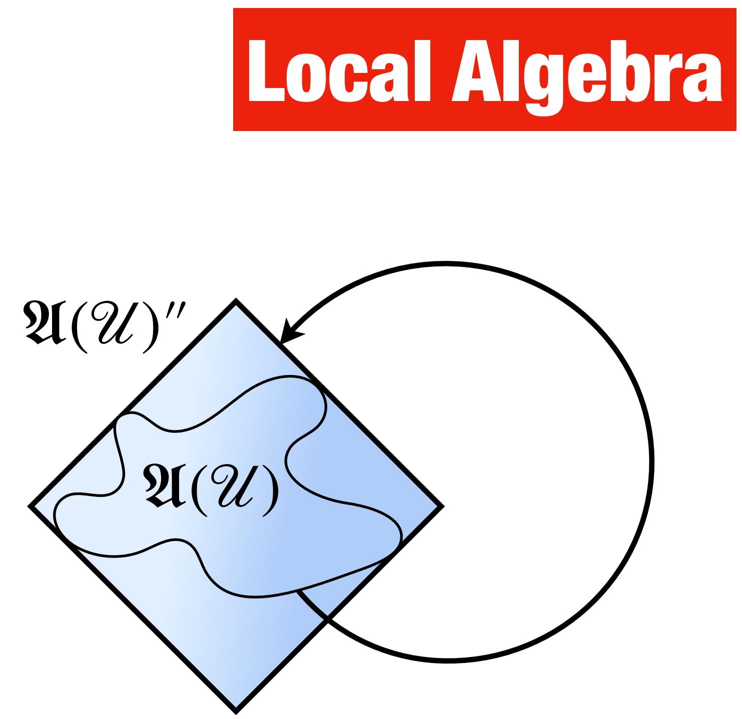
- Conclusion





$\mathfrak{A}(\mathscr{U})''$ is the smallest von Neumann algebra contains $\mathfrak{A}(\mathscr{U})$

$\mathfrak{A}$ is a von Neumann algebra $\Rightarrow \mathfrak{A} = \mathfrak{A}''$

THE REEH-SCHLIEDER THEOREM

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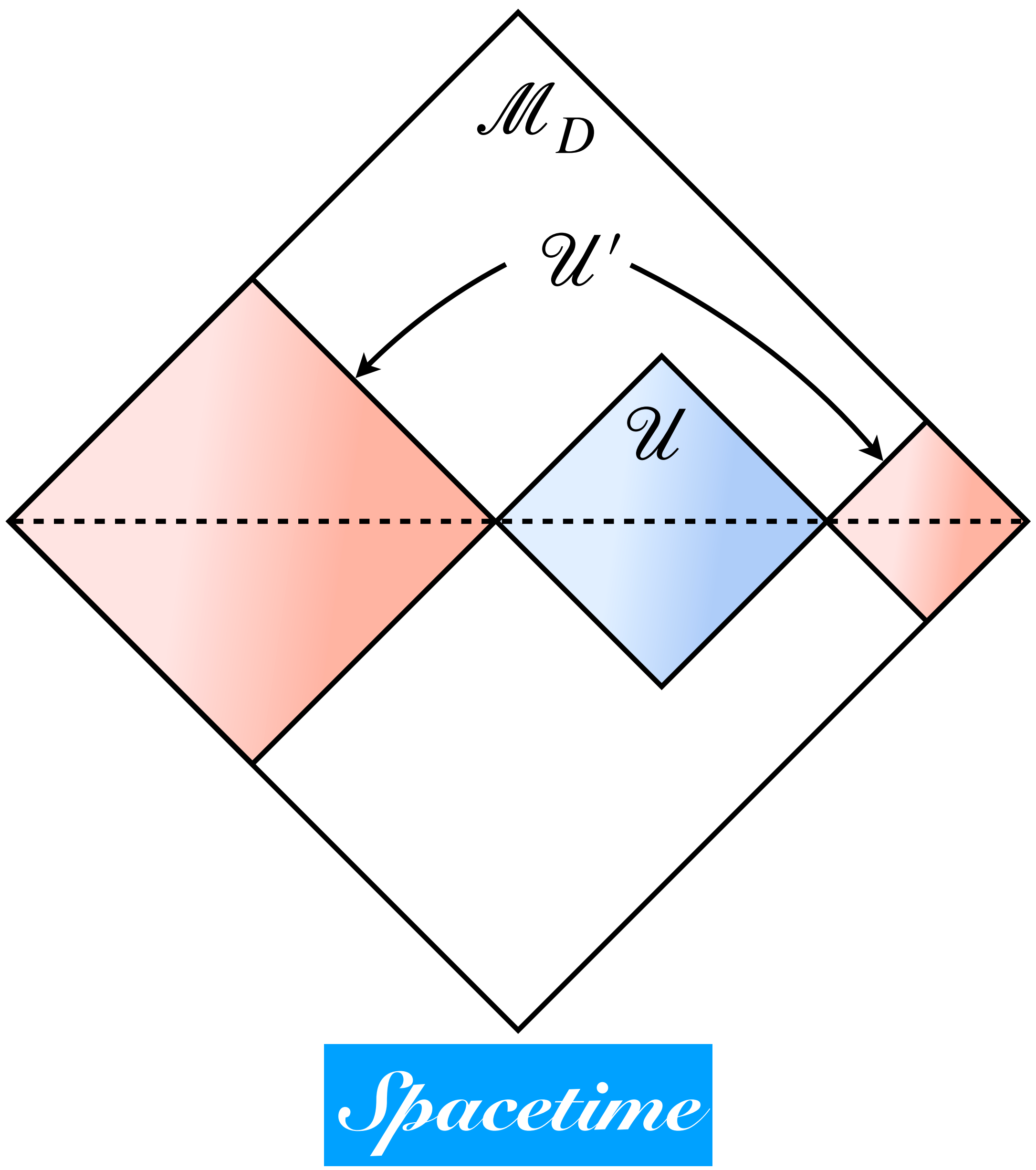
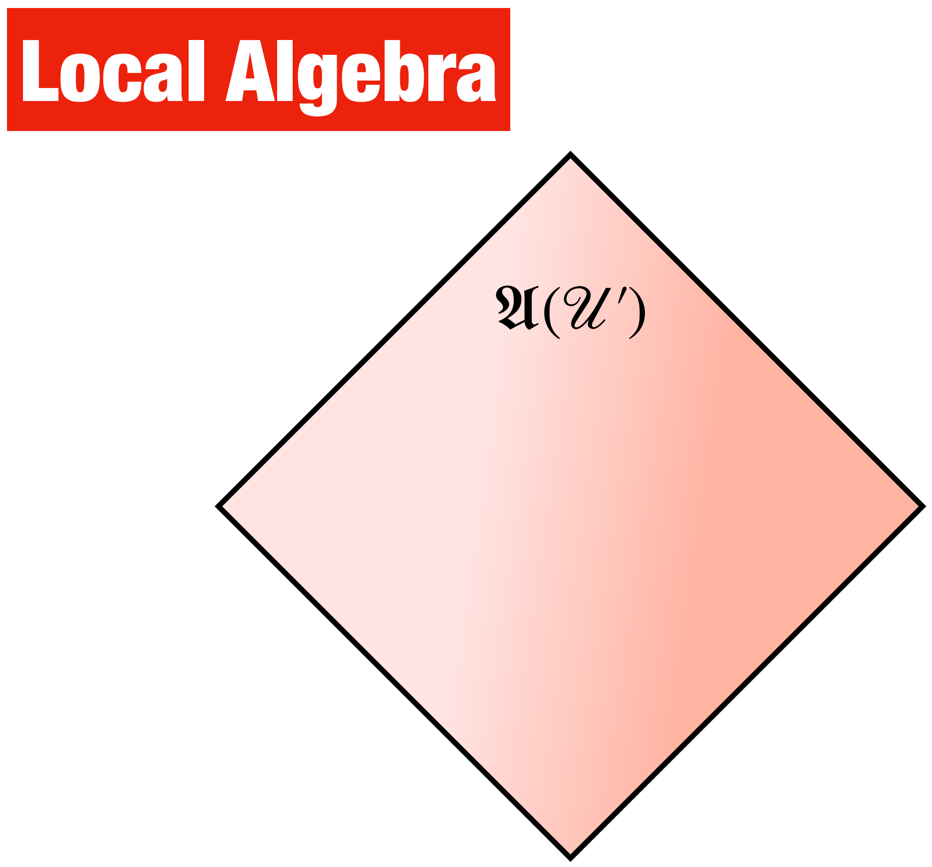
- Conclusion





THE REEH-SCHLIEDER THEOREM

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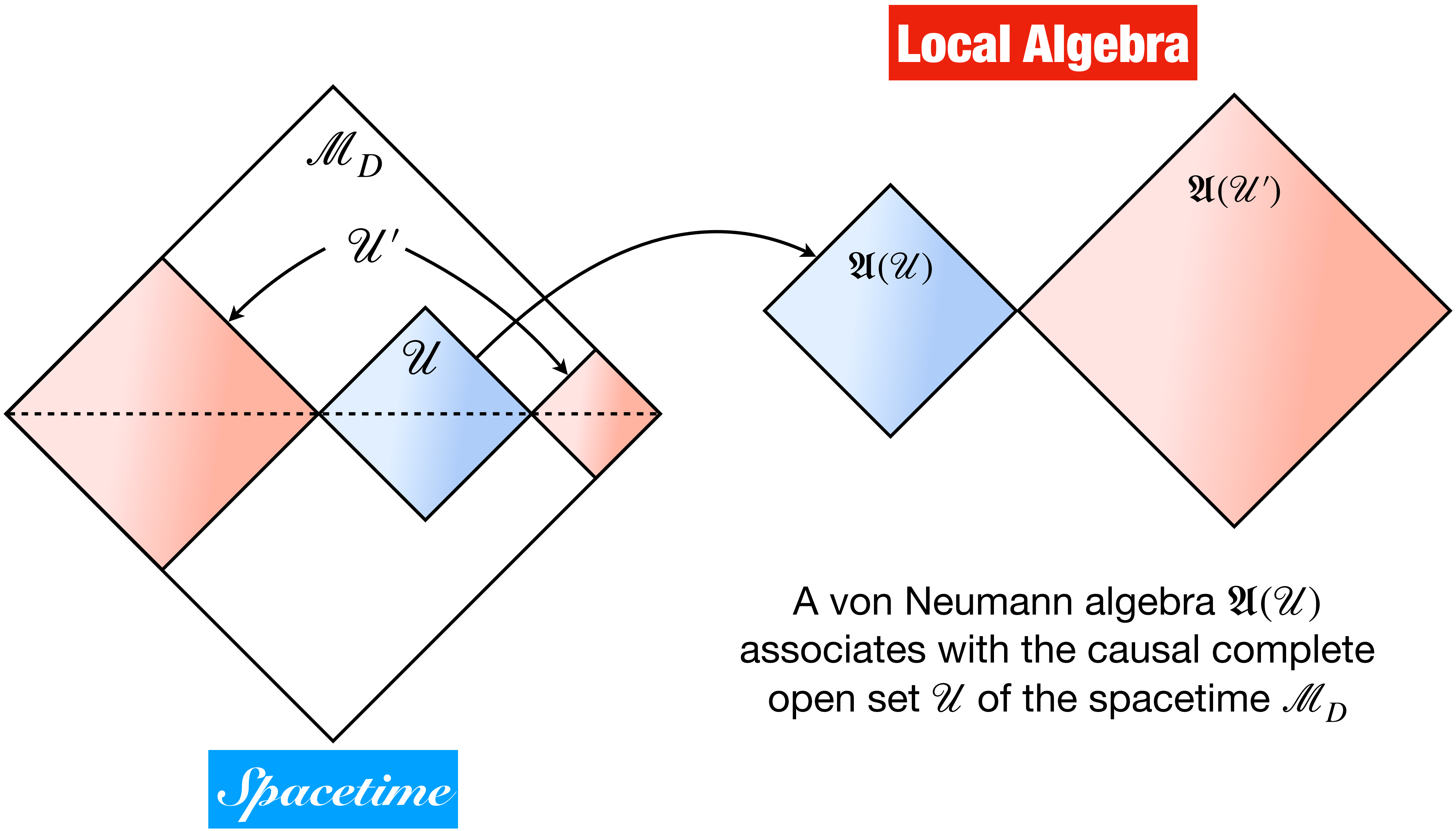

A von Neumann algebra $\mathfrak{A}(\mathscr{U})$ associates with the causal complete open set $\mathscr{U}$ of the spacetime $\mathscr{M}_D$

THE REEH-SCHLIEDER THEOREM

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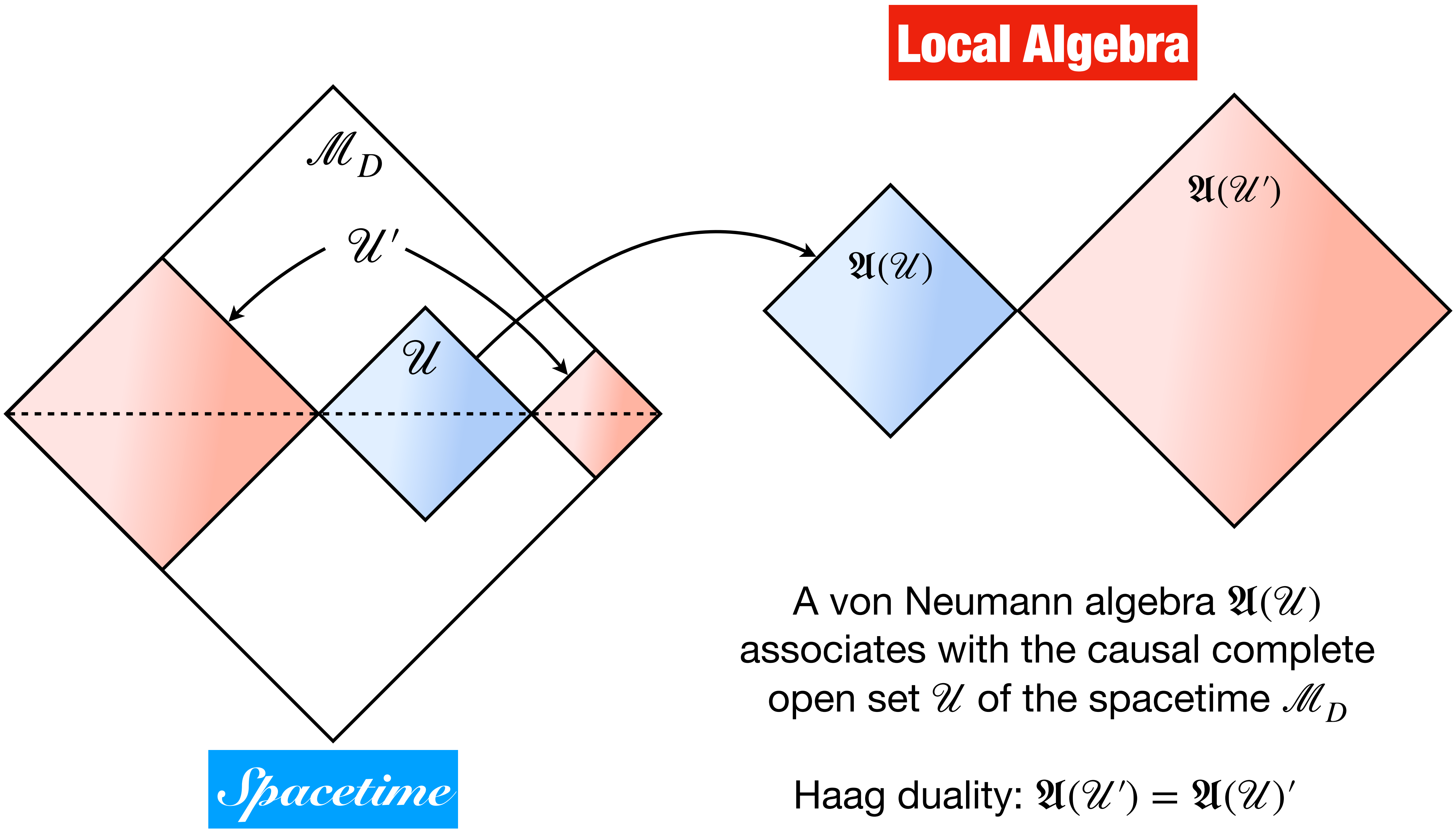
- Conclusion



A von Neumann algebra $\mathfrak{A}(\mathscr{U})$ associates with the causal complete open set $\mathscr{U}$ of the spacetime $\mathscr{M}_D$

Haag duality: $\mathfrak{A}(\mathscr{U}') = \mathfrak{A}(\mathscr{U})'$

THE REEH-SCHLIEDER THEOREM

Appendix (Hao)

- A Little about Algebraic Quantum Field Theory (AQFT)

- Definition

局域观测量公理（D 是有限的时空区域）

1. **单调性**（*Monotone property*）：若 $D_1 \supset D_2$，则 $\mathcal{O}(D_1) \supset \mathcal{O}(D_2)$。
2. **协变性**（*Covariance*）：对于 $g = (a, \Lambda) \in \mathscr{P}_+^\uparrow$，有 $\alpha_g \mathcal{O}(D) = \mathcal{O}(gD)$，其中 $gD = \{\Lambda x + a;\ x \in D\}$。
3. **局域性**（*Locality*）：如果 D_1 与 D_2 类空分隔，那么 $\mathcal{O}(D_1)$ 与 $\mathcal{O}(D_2)$ 对易。
4. **生成性**（*Generating property*）：$\cup_D \mathcal{O}(D)$ 生成作为观测量代数的 C^* 代数 $\mathfrak{A}$。

THE REEH-SCHLIEDER THEOREM

Appendix (Hao)

- A Little about Algebraic Quantum Field Theory (AQFT)
- Gelfand-Naimark–Segal representation

定理 2.18

对于 C^* 代数 $\mathfrak{A}$ 上的任意一个态 φ，都存在一个 Hilbert 空间 $\mathscr{H}_\varphi$，$\mathfrak{A}$ 在 $\mathscr{H}_\varphi$ 上的一个表示 π_φ，和 $\mathscr{H}_\varphi$ 中的一个单位矢量 Ω_φ，满足如下两个条件：

1. 对于任意的 $A \in \mathfrak{A}$

$$\varphi(A) = (\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi). \tag{2.36}$$

2. Ω_φ 是表示 π_φ 的一个**循环矢量**（*cyclic vector*），也就是说

$$\pi_\varphi(\mathfrak{A})\Omega_\varphi \equiv \{\pi_\varphi(A)\Omega_\varphi; \quad A \in \mathfrak{A}\}$$

在 $\mathscr{H}_\varphi$ 中是稠密的。

THE REEH-SCHLIEDER THEOREM

Appendix (Hao)

- A Little about Algebraic Quantum Field Theory (AQFT)
- Gelfand-Naimark–Segal representation

满足这两条的三元组 $(\mathscr{H}_\varphi, \pi_\varphi, \Omega_\varphi)$，作为幺正等价类是唯一的。因此，如果存在另一个由 Hilbert 空间 $\mathscr{H}'_\varphi$、$\mathfrak{A}$ 在 $\mathscr{H}'_\varphi$ 上的表示 π'_φ 和 $\mathscr{H}'_\varphi$ 中的矢量 Ω'_φ 组成的三元组，满足条件（1）和（2），则一定存在满足

$$U\pi_\varphi(A) = \pi'_\varphi(A)U \qquad \text{（对所有的元素 } A \in \mathfrak{A}) \tag{2.37}$$

$$U\Omega_\varphi = \Omega'_\varphi$$

的从 $\mathscr{H}_\varphi$ 到 $\mathscr{H}'_\varphi$ 的幺正映射 U。

THE REEH-SCHLIEDER THEOREM

Appendix (Hao)

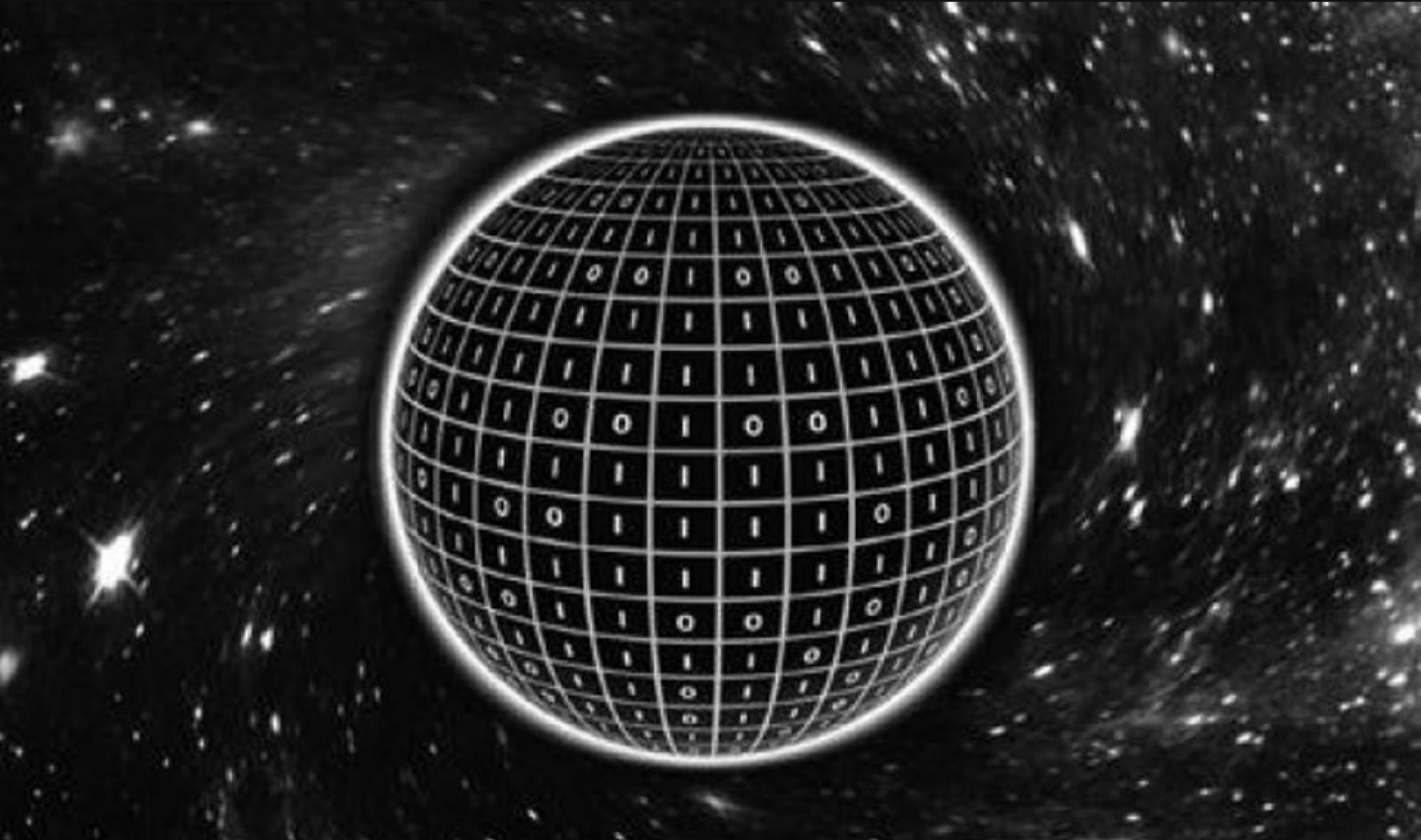
- A Little about Algebraic Quantum Field Theory (AQFT)
- Gelfand-Naimark–Segal representation

Israïl Moyseyovich Gel'fand
(1913/09/02-2009/10/05)

Mark Aronovich Naimark
(1909/12/05-1978/12/30)

Irving Ezra Segal
(1918/09/13-1998/08/30)

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY



THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- Tomita-Takesaki theory (富田-竹崎理论, 1967-1970)

Minoru Tomita
冨田 稔
(1924/02/06-2015/10/09)

Masamichi Takesaki
竹崎 正道
(1933/07/18-)

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- **The Tomita operator**:

let $|\Psi\rangle$ be a cyclic and separating vector (e.g., the vacuum vector) for the local observable algebra $\mathfrak{A}(\mathscr{U})$ and its commutant $\mathfrak{A}(\mathscr{U})'$, the Tomita operator for $|\Psi\rangle$ is an **antilinear** operator S_Ψ defined by

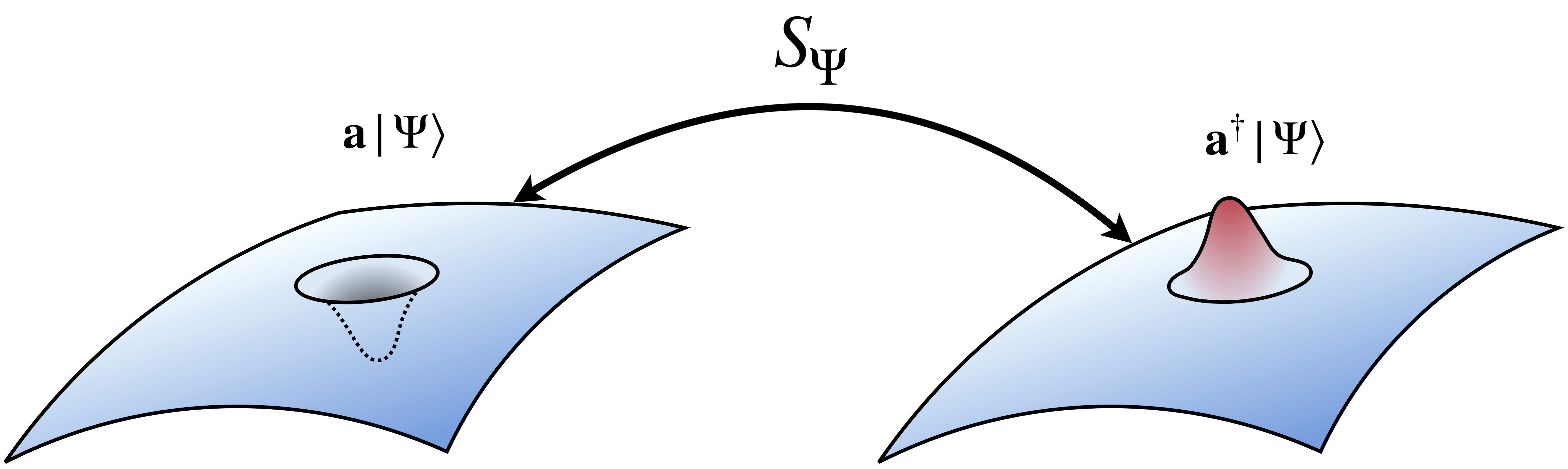
$$S_\Psi\,(\mathbf{a}\,|\,\Psi\rangle) = \mathbf{a}^\dagger\,|\,\Psi\rangle$$

for $\forall\ \mathbf{a} \in \mathfrak{A}(\mathscr{U})$.

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- **The Tomita operator**



$$S_\Psi\,(\mathbf{a}\,|\,\Psi\rangle) = \mathbf{a}^\dagger\,|\,\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- **The Tomita operator**

1. $|\Psi\rangle$ is separating $\Rightarrow$ $S_\Psi|0\rangle = 0$, the definition is consistent
2. $|\Psi\rangle$ is cyclic $\Rightarrow$ S_Ψ is defined on a dense subset of $\mathscr{H}$
3. (Closable) For $\mathbf{a}_n|\Psi\rangle \to x$, $\mathbf{a}_n \in \mathfrak{A}(\mathscr{U})$, if $\mathbf{a}_n^\dagger|\Psi\rangle \to y$ exists, we can extend the Tomita operator with $S_\Psi x = y$.
4. $S_\Psi^2(\mathbf{a}|\Psi\rangle) = S_\Psi(S_\Psi(\mathbf{a}|\Psi\rangle)) = S_\Psi(\mathbf{a}^\dagger|\Psi\rangle) = \mathbf{a}|\Psi\rangle \;\Rightarrow\; S_\Psi^2 = \mathbf{1}$
5. $S_\Psi|\Psi\rangle = |\Psi\rangle$

$$S_\Psi\,(\mathbf{a}|\Psi\rangle) = \mathbf{a}^\dagger|\Psi\rangle$$

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5. $S_\Psi|\Psi\rangle = |\Psi\rangle$

WARNING: I use the notation $|0\rangle$ for the zero vector in the Hilbert space, and $|\Omega\rangle$ for the vacuum vector. They are completely different concept.

$$S_\Psi\left(\mathbf{a}|\Psi\rangle\right) = \mathbf{a}^\dagger|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- The Tomita operator of $\mathfrak{A}(\mathscr{U})$ and $\mathfrak{A}(\mathscr{U})'$

$$S_\Psi\left(\mathbf{a}\,|\,\Psi\rangle\right) = \mathbf{a}^\dagger\,|\,\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- The Tomita operator of $\mathfrak{A}(\mathscr{U})$ and $\mathfrak{A}(\mathscr{U})'$

$$S'_\Psi\left(\mathbf{a}'|\Psi\rangle\right) = \mathbf{a}'^\dagger|\Psi\rangle$$

$$S_\Psi\left(\mathbf{a}|\Psi\rangle\right) = \mathbf{a}^\dagger|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

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$$S'_\Psi\ (\mathbf{a}'\,|\,\Psi\rangle) = \mathbf{a}'^\dagger\,|\,\Psi\rangle$$

$$\langle \mathbf{a}\Psi\,|\,S'_\Psi \mathbf{a}'\Psi\rangle = \langle\Psi\,|\,\mathbf{a}^\dagger \mathbf{a}'^\dagger\,|\,\Psi\rangle = \langle\Psi\,|\,\mathbf{a}'^\dagger \mathbf{a}^\dagger\,|\,\Psi\rangle = \langle \mathbf{a}'\Psi\,|\,S_\Psi \mathbf{a}\Psi\rangle = \overline{\langle S_\Psi \mathbf{a}\Psi\,|\,\mathbf{a}'\Psi\rangle}$$

$$S_\Psi\ (\mathbf{a}\,|\,\Psi\rangle) = \mathbf{a}^\dagger\,|\,\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- The Tomita operator of $\mathfrak{A}(\mathscr{U})$ and $\mathfrak{A}(\mathscr{U})'$

$$S'_\Psi\ (\mathbf{a}'|\Psi\rangle) = \mathbf{a}'^\dagger|\Psi\rangle$$

$$\langle \mathbf{a}\Psi | S'_\Psi \mathbf{a}'\Psi\rangle = \langle \Psi | \mathbf{a}^\dagger \mathbf{a}'^\dagger | \Psi\rangle = \langle \Psi | \mathbf{a}'^\dagger \mathbf{a}^\dagger | \Psi\rangle = \langle \mathbf{a}'\Psi | S_\Psi \mathbf{a}\Psi\rangle = \overline{\langle S_\Psi \mathbf{a}\Psi | \mathbf{a}'\Psi\rangle}$$

$$\Rightarrow \quad S'_\Psi = S^\dagger_\Psi$$

$$S_\Psi\ (\mathbf{a}|\Psi\rangle) = \mathbf{a}^\dagger|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- The Tomita operator of $\mathfrak{A}(\mathscr{U})$ and $\mathfrak{A}(\mathscr{U})'$

$$S'_\Psi \left(\mathbf{a}' | \Psi\rangle\right) = \mathbf{a}'^\dagger | \Psi\rangle$$

$$\langle \mathbf{a}\Psi | S'_\Psi \mathbf{a}'\Psi\rangle = \langle \Psi | \mathbf{a}^\dagger \mathbf{a}'^\dagger | \Psi\rangle = \langle \Psi | \mathbf{a}'^\dagger \mathbf{a}^\dagger | \Psi\rangle = \langle \mathbf{a}'\Psi | S_\Psi \mathbf{a}\Psi\rangle = \overline{\langle S_\Psi \mathbf{a}\Psi | \mathbf{a}'\Psi\rangle}$$

$\Rightarrow \; S'_\Psi = S^\dagger_\Psi$ (in their domain!)

$$S_\Psi \left(\mathbf{a} | \Psi\rangle\right) = \mathbf{a}^\dagger | \Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- **The polar decomposition theorem**: if A is a closed, densely defined unbounded operator between complex Hilbert spaces, then is has a (unique) polar decomposition $A = U|A|$, where $|A|$ is a (possibly unbounded) non-negative self-adjoint operator with the same domain as A, and U is a partial isometry vanishing on the orthogonal complement of the range Ran$(|A|)$.

- So the Tomita operator has a unique polar decomposition $S_\Psi = J_\Psi \Delta_\Psi^{1/2}$, where J_Ψ is antiunitary and $\Delta_\Psi^{1/2}$ is Hermitian and positive definite.

- $\Delta_\Psi = S_\Psi^\dagger S_\Psi$

$$S_\Psi \left(\mathbf{a}\,|\,\Psi\rangle\right) = \mathbf{a}^\dagger\,|\,\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- The Δ_Ψ is called the **modular operator** (usually unbounded)
- The J_Ψ is called the **modular conjugation**
- $S_\Psi|\Psi\rangle = S_\Psi^\dagger|\Psi\rangle = |\Psi\rangle \;\Rightarrow\; \Delta_\Psi|\Psi\rangle = |\Psi\rangle$, for any function f, $f(\Delta_\Psi)|\Psi\rangle = f(1)|\Psi\rangle$.

$$S_\Psi\left(\mathbf{a}|\Psi\rangle\right) = \mathbf{a}^\dagger|\Psi\rangle$$

THE MODULAR OPERATOR AND RELATIVE ENTROPY IN QUANTUM FIELD THEOTY

I. Definition and first properties

- **The modular operator and modular conjugation**

1. $S_\Psi^2 = \mathbf{1} \;\Rightarrow\; J_\Psi \Delta_\Psi^{1/2} J_\Psi \Delta_\Psi^{1/2} = \mathbf{1} \;\Rightarrow\; J_\Psi \Delta_\Psi^{1/2} J_\Psi = \Delta_\Psi^{-1/2}$

2. $J_\Psi^2 (J_\Psi^{-1} \Delta_\Psi^{1/2} J_\Psi) = \Delta_\Psi^{-1/2} = \mathbf{1} \cdot \Delta_\Psi^{-1/2}$. So they are both polar decomposition of $\Delta_\Psi^{-1/2}$. By the uniqueness of the polar decomposition, $J_\Psi^2 = \mathbf{1}$

3. $S'_\Psi = S_\Psi^\dagger = \Delta_\Psi^{1/2} J_\Psi = J_\Psi \Delta_\Psi^{-1/2}$, so $J'_\Psi = J_\Psi,\ \Delta'_\Psi = \Delta_\Psi^{-1}$

4. Because $J_\Psi \Delta_\Psi J_\Psi = \Delta_\Psi^{-1}$, we have $J_\Psi f(\Delta_\Psi) J_\Psi = \bar{f}(\Delta_\Psi^{-1})$ for any function f

5. For example, when $f(z) = z^{is}\ (s \in \mathbb{R})$, we have $J_\Psi \Delta_\Psi^{is} J_\Psi = \Delta_\Psi^{is}\ (s \in \mathbb{R})$

$S_\Psi\,(\mathbf{a}\,|\,\Psi\rangle) = \mathbf{a}^\dagger\,|\,\Psi\rangle$


To Be Continued...