Entanglement properties of quantum field theory

A note of Witten's paper "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory"

Part III: From Finite-dimensional Quantum Systems and Some Lessons to A Fundamental Example in Quantum Field Theory

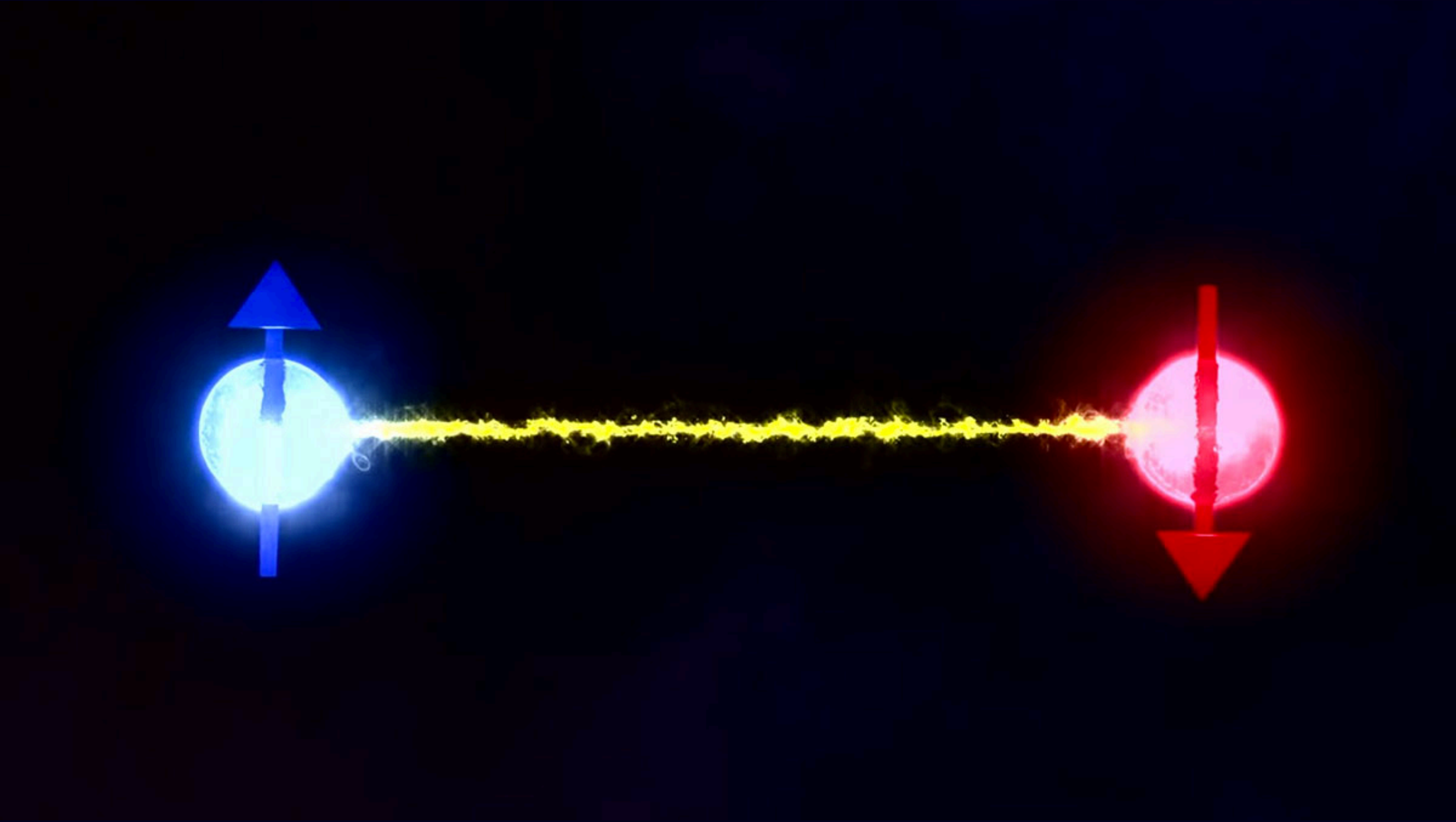
Hao Zhang

Theoretical Physics Division, Institute of High Energy Physics, Chinese Academy of Sciences

A Review

- The Reeh-Schlieder Theorem
- The Modular Operator and Relative Entropy

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

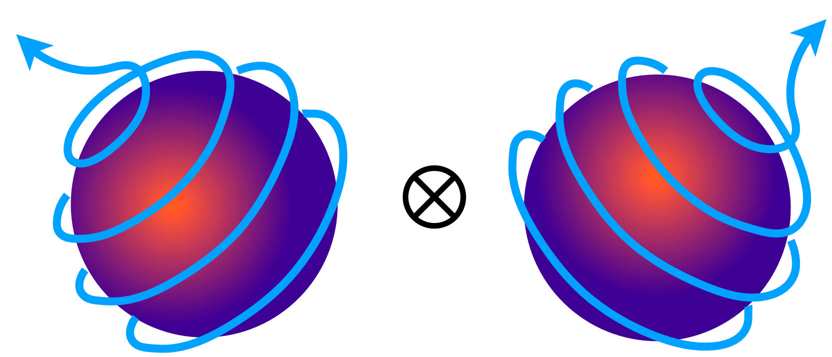
I. The modular operators in the finite-dimensional case

- Bipartite quantum system

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$$

$$\mathbf{a} : \; \mathscr{H}_1 \to \mathscr{H}_1, \mathbf{a} \in \mathfrak{A} \qquad \mathbf{a}' : \; \mathscr{H}_2 \to \mathscr{H}_2, \mathbf{a} \in \mathfrak{A}'$$

$$\mathbf{a} \otimes \mathbf{1} : \; \mathscr{H} \to \mathscr{H}, \qquad \mathbf{1} \otimes \mathbf{a}' : \; \mathscr{H} \to \mathscr{H}$$



$$|\psi_1\rangle \quad \otimes \quad |\psi_2\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Cyclic and separating vectors for $\mathfrak{A}$ and $\mathfrak{A}'$
  - SVD theorem $\Rightarrow$ $\forall\ \Psi \in \mathscr{H}$, one could find out suitable orthonormal bases $\{\psi_i\}$ of $\mathscr{H}_1$ and $\{\varphi_j\}$ of $\mathscr{H}_2$, which give

$$\Psi = \sum_k c_k |\psi_k\rangle \otimes |\varphi_k\rangle \equiv \sum_k c_k |k, k\rangle$$

  - A (linear) operator $\mathbf{a}$ in $\mathfrak{A}$ acts on Ψ as

$$(\mathbf{a} \otimes \mathbf{1})\Psi = \sum_k c_k \mathbf{a} |k, k\rangle \equiv \sum_k c_k a_{jk} |j, k\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Cyclic and separating vectors for $\mathfrak{A}$ and $\mathfrak{A}'$
  - A (linear) operator $\mathbf{a}$ in $\mathfrak{A}$ acts on Ψ as

$$(\mathbf{a} \otimes \mathbf{1})\Psi = \sum_k c_k \mathbf{a} \,|\, k, k\rangle \equiv \sum_k c_k a_{jk} \,|\, j, k\rangle$$

$$A_{\dim \mathscr{H}_1 \times \dim \mathscr{H}_1} C_{\dim \mathscr{H}_1 \times \dim \mathscr{H}_2} = (AC)_{\dim \mathscr{H}_1 \times \dim \mathscr{H}_2}$$

  - Ψ is cyclic $\Rightarrow$ for any $\dim \mathscr{H}_1 \times \dim \mathscr{H}_2$ matrix M, equation $AC = M$ has solution, so C must be full-rank and $\dim \mathscr{H}_1 \geqslant \dim \mathscr{H}_2$.
  - Ψ is separating $\Rightarrow$ $A = 0$ is the unique solution of equation $AC = 0$, so C must be full-rank and $\dim \mathscr{H}_1 \leqslant \dim \mathscr{H}_2$.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Cyclic and separating vectors for $\mathfrak{A}$ and $\mathfrak{A}'$
    - Ψ is a cyclic and separating vector of $\mathfrak{A}$ and $\mathfrak{A}'$, iff for suitable orthonormal bases $\{\psi_i\}$ of $\mathscr{H}_1$ and $\{\varphi_j\}$ of $\mathscr{H}_2$,

$$\Psi = \sum_k c_k |\psi_k\rangle \otimes |\varphi_k\rangle \equiv \sum_k c_k |k, k\rangle$$

    and $c_k \neq 0$ for all $k = 1, \cdots, \dim \mathscr{H}_1$, and $\dim \mathscr{H}_1 = \dim \mathscr{H}_2$.

    - Or equivalently, C is a non-degenerate diagonal square matrix.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi$$

- Consider n^2 operators (matrices) $\mathbf{a}[ij],\ i, j = 1,\cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$\mathbf{a}[ij]\,|\,l,\, k\rangle = \delta_{li}\,|\,j,\, k\rangle \quad (\mathbf{a}[ij]^\dagger\,|\,l,\, k\rangle = \delta_{jl}\,|\,i,\, k\rangle)$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi$$

- Consider n^2 operators (matrices) $\mathbf{a}[ij],\ i, j = 1, \cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$\mathbf{a}[ij]\,|\,l,\, k\rangle = \delta_{li}\,|\,j,\, k\rangle \quad (\mathbf{a}[ij]^\dagger\,|\,l,\, k\rangle = \delta_{jl}\,|\,i,\, k\rangle)$$

$$A[ij] = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi$$

- Consider n^2 operators (matrices) $\mathbf{a}[ij],\ i, j = 1, \cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$\mathbf{a}[ij]\,|\,l,\, k\rangle = \delta_{li}\,|\,j,\, k\rangle \quad (\mathbf{a}[ij]^\dagger\,|\,l,\, k\rangle = \delta_{jl}\,|\,i,\, k\rangle)$$

$$A[ij] = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

the j-th row

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi$$

- Consider n^2 operators (matrices) $\mathbf{a}[ij],\ i, j = 1, \cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$\mathbf{a}[ij]\,|\,l, k\rangle = \delta_{li}\,|\,j, k\rangle \quad (\mathbf{a}[ij]^\dagger\,|\,l, k\rangle = \delta_{jl}\,|\,i, k\rangle)$$

$$A[ij] = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

the j-th row

the i-th column

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi$$

- Consider n^2 operators (matrices) $\mathbf{a}[ij],\ i, j = 1,\cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$S_\Psi((\mathbf{a}[ij] \otimes \mathbf{1})\Psi) = (\mathbf{a}[ij]^\dagger \otimes \mathbf{1})\Psi$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi$$

- Consider n^2 operators (matrices) $\mathbf{a}[ij],\ i, j = 1,\cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$S_\Psi\left((\mathbf{a}[ij] \otimes \mathbf{1}) \sum_k c_k |k,\ k\rangle \right) = (\mathbf{a}[ij]^\dagger \otimes \mathbf{1}) \sum_k c_k |k,\ k\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi$$

- Consider n^2 operators (matrices) $\mathbf{a}[ij],\ i, j = 1, \cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$S_\Psi\left((\mathbf{a}[ij] \otimes \mathbf{1}) \sum_k c_k |k,\, k\rangle \right) = (\mathbf{a}[ij]^\dagger \otimes \mathbf{1}) \sum_k c_k |k,\, k\rangle$$

$$S_\Psi\left(c_i |j,\, i\rangle\right) = c_j |i,\, j\rangle \;\;\Rightarrow\;\; \bar{c}_i\, S_\Psi |j,\, i\rangle = c_j |i,\, j\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi$$

- Consider n^2 operators (matrices) $\mathbf{a}[ij],\ i, j = 1,\cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$S_\Psi\left((\mathbf{a}[ij] \otimes \mathbf{1}) \sum_k c_k |k,\, k\rangle \right) = (\mathbf{a}[ij]^\dagger \otimes \mathbf{1}) \sum_k c_k |k,\, k\rangle$$

$$S_\Psi \left(c_i |j,\, i\rangle \right) = c_j |i,\, j\rangle \ \Rightarrow\ \bar{c}_i\, S_\Psi |j,\, i\rangle = c_j |i,\, j\rangle$$

$$S_\Psi |j,\, i\rangle = \frac{c_j}{\bar{c}_i} |i,\, j\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi |j,\, i\rangle = \frac{c_j}{\bar{c}_i} |i,\, j\rangle$$

- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi |j,\, i\rangle = \frac{c_j}{\bar{c}_i} |i,\, j\rangle$$

- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

$$\langle k,\, l | \left(S_\Psi^\dagger |j,\, i\rangle \right) = \langle j,\, i | (S_\Psi |k,\, l\rangle) = \langle j,\, i | \left(\frac{c_k}{\bar{c}_l} |l,\, k\rangle \right) = \frac{c_k}{\bar{c}_l} \langle j,\, i | l,\, k\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi |j, i\rangle = \frac{c_j}{\bar{c}_i} |i, j\rangle$$

- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

$$\langle k, l | \left(S_\Psi^\dagger |j, i\rangle \right) = \langle j, i | (S_\Psi |k, l\rangle) = \langle j, i | \left(\frac{c_k}{\bar{c}_l} |l, k\rangle \right) = \frac{c_k}{\bar{c}_l} \langle j, i | l, k\rangle$$

$$\Rightarrow \quad S_\Psi^\dagger |j, i\rangle = \frac{c_i}{\bar{c}_j} |i, j\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi |j, i\rangle = \frac{c_j}{\bar{c}_i} |i, j\rangle$$

- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

$$\langle k, l | \left(S_\Psi^\dagger |j, i\rangle \right) = \langle j, i | (S_\Psi |k, l\rangle) = \langle j, i | \left(\frac{c_k}{\bar{c}_l} |l, k\rangle \right) = \frac{c_k}{\bar{c}_l} \langle j, i | l, k\rangle$$

$$\Rightarrow \quad S_\Psi^\dagger |j, i\rangle = \frac{c_i}{\bar{c}_j} |i, j\rangle$$

$$\Delta_\Psi |j, i\rangle = S_\Psi^\dagger S_\Psi |j, i\rangle = S_\Psi^\dagger \left(\frac{c_j}{\bar{c}_i} |i, j\rangle \right) = \frac{\bar{c}_j}{c_i} S_\Psi^\dagger |i, j\rangle = \frac{\bar{c}_j c_j}{c_i \bar{c}_i} |j, i\rangle = \frac{|c_j|^2}{|c_i|^2} |j, i\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi |j, i\rangle = \frac{c_j}{\bar{c}_i} |i, j\rangle$$

- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

$$\Delta_\Psi^{1/2} |j, i\rangle = \sqrt{\frac{|c_j|^2}{|c_i|^2}} |j, i\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi |j, i\rangle = \frac{c_j}{\bar{c}_i} |i, j\rangle$$

- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi |j, i\rangle = \frac{c_j}{\bar{c}_i} |i, j\rangle$$

- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

$$\frac{c_j}{\bar{c}_i} |i, j\rangle = S_\Psi |j, i\rangle = J_\Psi \Delta_\Psi^{1/2} |j, i\rangle = \sqrt{\frac{|c_j|^2}{|c_i|^2}} J_\Psi |j, i\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi | j, i\rangle = \frac{c_j}{\bar{c}_i} | i, j\rangle$$

- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

$$\frac{c_j}{\bar{c}_i} | i, j\rangle = S_\Psi | j, i\rangle = J_\Psi \Delta_\Psi^{1/2} | j, i\rangle = \sqrt{\frac{|c_j|^2}{|c_i|^2}} J_\Psi | j, i\rangle$$

$$\therefore \; J_\Psi | j, i\rangle = \frac{c_j}{\bar{c}_i} \sqrt{\frac{c_i \bar{c}_i}{c_j \bar{c}_j}} | i, j\rangle = \sqrt{\frac{c_i c_j}{\bar{c}_i \bar{c}_j}} | i, j\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The Tomita operator $S_\Psi : \mathscr{H} \to \mathscr{H}$

$$S_\Psi((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Psi \qquad S_\Psi |j, i\rangle = \frac{c_j}{\bar{c}_i} |i, j\rangle$$

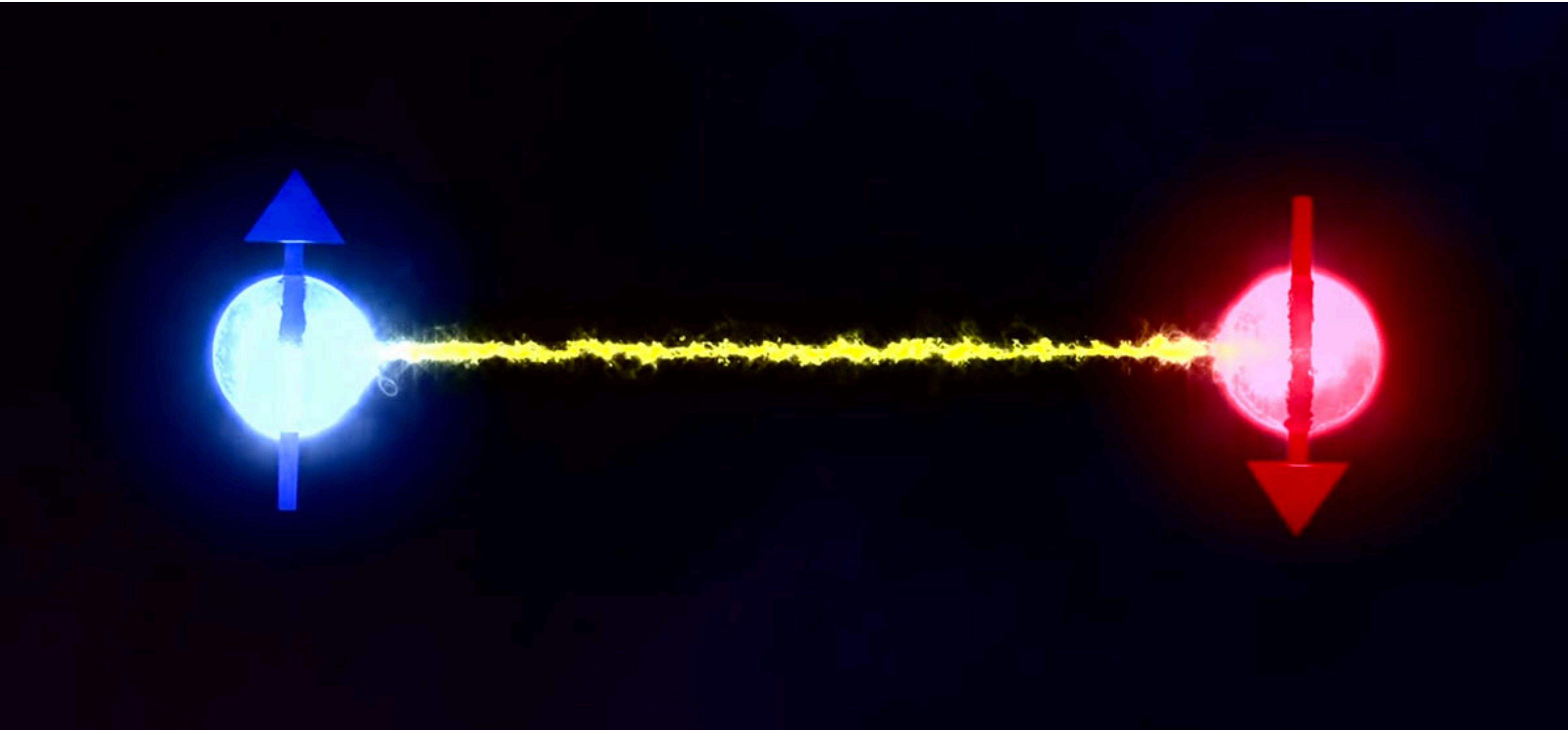
- The modular operator $\Delta_\Psi : \mathscr{H} \to \mathscr{H}$ and modular conjugation $J_\Psi : \mathscr{H} \to \mathscr{H}$

$$\Delta_\Psi |j, i\rangle = \frac{|c_j|^2}{|c_i|^2} |j, i\rangle, \quad J_\Psi |j, i\rangle = \sqrt{\frac{c_i c_j}{\bar{c}_i \bar{c}_j}} |i, j\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle

$$\Psi_1 = \frac{1}{\sqrt{2}} | \uparrow \uparrow \rangle + \frac{1}{\sqrt{2}} | \downarrow \downarrow \rangle, \quad \Rightarrow \; c_\uparrow = \frac{1}{\sqrt{2}}, c_\downarrow = \frac{1}{\sqrt{2}}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle

$$\Psi_1 = \frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\rangle, \quad \Rightarrow \; c_\uparrow = \frac{1}{\sqrt{2}}, c_\downarrow = \frac{1}{\sqrt{2}}$$

$$S_\Psi|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle, S_\Psi|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle, S_\Psi|\downarrow\uparrow\rangle = |\uparrow\downarrow\rangle, S_\Psi|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle

$$\Psi_1 = \frac{1}{\sqrt{2}} | \uparrow \uparrow \rangle + \frac{1}{\sqrt{2}} | \downarrow \downarrow \rangle, \quad \Rightarrow \; c_\uparrow = \frac{1}{\sqrt{2}}, c_\downarrow = \frac{1}{\sqrt{2}}$$

$$S_\Psi | \uparrow \uparrow \rangle = | \uparrow \uparrow \rangle, S_\Psi | \uparrow \downarrow \rangle = | \downarrow \uparrow \rangle, S_\Psi | \downarrow \uparrow \rangle = | \uparrow \downarrow \rangle, S_\Psi | \downarrow \downarrow \rangle = | \downarrow \downarrow \rangle$$

$$\Delta_\Psi | \uparrow \uparrow \rangle = | \uparrow \uparrow \rangle, \Delta_\Psi | \uparrow \downarrow \rangle = | \uparrow \downarrow \rangle, \Delta_\Psi | \downarrow \uparrow \rangle = | \downarrow \uparrow \rangle, \Delta_\Psi | \downarrow \downarrow \rangle = | \downarrow \downarrow \rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle

$$\Psi_1 = \frac{1}{\sqrt{2}} | \uparrow \uparrow \rangle + \frac{1}{\sqrt{2}} | \downarrow \downarrow \rangle, \quad \Rightarrow \; c_\uparrow = \frac{1}{\sqrt{2}}, \, c_\downarrow = \frac{1}{\sqrt{2}}$$

$$S_\Psi | \uparrow \uparrow \rangle = | \uparrow \uparrow \rangle, \, S_\Psi | \uparrow \downarrow \rangle = | \downarrow \uparrow \rangle, \, S_\Psi | \downarrow \uparrow \rangle = | \uparrow \downarrow \rangle, \, S_\Psi | \downarrow \downarrow \rangle = | \downarrow \downarrow \rangle$$

$$\Delta_\Psi | \uparrow \uparrow \rangle = | \uparrow \uparrow \rangle, \, \Delta_\Psi | \uparrow \downarrow \rangle = | \uparrow \downarrow \rangle, \, \Delta_\Psi | \downarrow \uparrow \rangle = | \downarrow \uparrow \rangle, \, \Delta_\Psi | \downarrow \downarrow \rangle = | \downarrow \downarrow \rangle$$

$$J_\Psi | \uparrow \uparrow \rangle = | \uparrow \uparrow \rangle, \, J_\Psi | \uparrow \downarrow \rangle = | \downarrow \uparrow \rangle, \, J_\Psi | \downarrow \uparrow \rangle = | \uparrow \downarrow \rangle, \, J_\Psi | \downarrow \downarrow \rangle = | \downarrow \downarrow \rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle

$$\Psi_2 = \cos\theta | \uparrow \uparrow \rangle + e^{i\varphi} \sin\theta | \downarrow \downarrow \rangle, \; \theta \in (0, \pi/2), \quad \Rightarrow \; c_\uparrow = \cos\theta, \; c_\downarrow = e^{i\varphi}\sin\theta$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle

$$\Psi_2 = \cos\theta | \uparrow\uparrow \rangle + e^{i\varphi} \sin\theta | \downarrow\downarrow \rangle, \, \theta \in (0, \, \pi/2), \quad \Rightarrow \;\; c_\uparrow = \cos\theta, \, c_\downarrow = e^{i\varphi}\sin\theta$$

$$S_\Psi | \uparrow\uparrow \rangle = | \uparrow\uparrow \rangle, \, S_\Psi | \uparrow\downarrow \rangle = e^{i\varphi} \cot\theta | \downarrow\uparrow \rangle, \, S_\Psi | \downarrow\uparrow \rangle = e^{i\varphi} \tan\theta | \uparrow\downarrow \rangle,$$

$$S_\Psi | \downarrow\downarrow \rangle = e^{2i\varphi} | \downarrow\downarrow \rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle

$$\Psi_2 = \cos\theta |\uparrow\uparrow\rangle + e^{i\varphi}\sin\theta |\downarrow\downarrow\rangle,\ \theta \in (0,\ \pi/2), \quad \Rightarrow\ c_\uparrow = \cos\theta,\ c_\downarrow = e^{i\varphi}\sin\theta$$

$$S_\Psi |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle,\ S_\Psi |\uparrow\downarrow\rangle = e^{i\varphi}\cot\theta |\downarrow\uparrow\rangle,\ S_\Psi |\downarrow\uparrow\rangle = e^{i\varphi}\tan\theta |\uparrow\downarrow\rangle,$$

$$S_\Psi |\downarrow\downarrow\rangle = e^{2i\varphi} |\downarrow\downarrow\rangle$$

$$\Delta_\Psi |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle,\ \Delta_\Psi |\uparrow\downarrow\rangle = \cot^2\theta |\uparrow\downarrow\rangle,\ \Delta_\Psi |\downarrow\uparrow\rangle = \tan^2\theta |\downarrow\uparrow\rangle,$$

$$\Delta_\Psi |\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- Example: two qubits system by a pair of spin-1/2 particle

$$\Psi_2 = \cos\theta|\uparrow\uparrow\rangle + e^{i\varphi}\sin\theta|\downarrow\downarrow\rangle,\ \theta\in(0,\ \pi/2), \quad \Rightarrow\ c_\uparrow = \cos\theta,\ c_\downarrow = e^{i\varphi}\sin\theta$$

$$S_\Psi|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle,\ S_\Psi|\uparrow\downarrow\rangle = e^{i\varphi}\cot\theta|\downarrow\uparrow\rangle,\ S_\Psi|\downarrow\uparrow\rangle = e^{i\varphi}\tan\theta|\uparrow\downarrow\rangle,$$

$$S_\Psi|\downarrow\downarrow\rangle = e^{2i\varphi}|\downarrow\downarrow\rangle$$

$$\Delta_\Psi|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle,\ \Delta_\Psi|\uparrow\downarrow\rangle = \cot^2\theta|\uparrow\downarrow\rangle,\ \Delta_\Psi|\downarrow\uparrow\rangle = \tan^2\theta|\downarrow\uparrow\rangle,$$

$$\Delta_\Psi|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle$$

$$J_\Psi|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle,\ J_\Psi|\uparrow\downarrow\rangle = e^{i\varphi}|\downarrow\uparrow\rangle,\ J_\Psi|\downarrow\uparrow\rangle = e^{i\varphi}|\uparrow\downarrow\rangle,\ J_\Psi|\downarrow\downarrow\rangle = e^{2i\varphi}|\downarrow\downarrow\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|k, k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\alpha, \alpha\rangle$$

- Consider n^2 operators (matrices) $\mathbf{a}[i\alpha],\ i, \alpha = 1, \cdots, n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$\mathbf{a}[i\alpha]\,|\,l,\,k\rangle = \delta_{li}\,|\,\alpha,\,k\rangle \quad (\mathbf{a}[i\alpha]^\dagger\,|\,\mu,\,\nu\rangle = \delta_{\alpha\mu}\,|\,i,\,\nu\rangle)$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|k,\,k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\alpha,\,\alpha\rangle$$

- Consider n^2 operators (matrices) $\mathbf{a}[i\alpha]$, $i,\alpha = 1,\cdots,n = \dim\mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$\mathbf{a}[i\alpha]\,|\,l,\,k\rangle = \delta_{li}\,|\,\alpha,\,k\rangle \quad (\mathbf{a}[i\alpha]^\dagger\,|\,\mu,\,\nu\rangle = \delta_{\alpha\mu}\,|\,i,\,\nu\rangle)$$

$$A[i\alpha] = \begin{pmatrix} 0 & 0 & \langle 1\,|\,\alpha\rangle & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \langle i\,|\,\alpha\rangle & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \langle n\,|\,\alpha\rangle & \cdots & \cdots & 0 \end{pmatrix}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k|k, k\rangle, \quad \Phi = \sum_\alpha d_\alpha|\alpha, \alpha\rangle$$

- Consider n^2 operators (matrices) $\mathbf{a}[i\alpha],\ i, \alpha = 1,\cdots, n = \dim\mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$\mathbf{a}[i\alpha]\,|\,l,\, k\rangle = \delta_{li}\,|\,\alpha,\, k\rangle \quad (\mathbf{a}[i\alpha]^\dagger\,|\,\mu,\, \nu\rangle = \delta_{\alpha\mu}\,|\,i,\, \nu\rangle)$$

$$A[i\alpha] = \begin{pmatrix} 0 & 0 & \langle 1|\alpha\rangle & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \langle i|\alpha\rangle & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \langle n|\alpha\rangle & \cdots & \cdots & 0 \end{pmatrix}$$

the i-th column

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi)=(\mathbf{a}^{\dagger}\otimes\mathbf{1})\Phi,\ \ \Psi=\sum_k c_k\,|k,\,k\rangle,\ \ \Phi=\sum_\alpha d_\alpha\,|\alpha,\,\alpha\rangle$$

- Consider n^2 operators (matrices) $\mathbf{a}[i\alpha],\ i,\alpha=1,\cdots,n=\dim\mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$S_{\Psi|\Phi}((\mathbf{a}[i\alpha]\otimes\mathbf{1})\Psi)=(\mathbf{a}[i\alpha]^{\dagger}\otimes\mathbf{1})\Phi$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi,\ \ \Psi = \sum_k c_k|k,\, k\rangle,\ \ \Phi = \sum_\alpha d_\alpha|\alpha,\, \alpha\rangle$$

- Consider n^2 operators (matrices) $\mathbf{a}[i\alpha],\ i,\alpha = 1,\cdots,n = \dim \mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$S_{\Psi|\Phi}\left((\mathbf{a}[i\alpha]\otimes\mathbf{1})\sum_k c_k|k,\, k\rangle\right) = (\mathbf{a}[i\alpha]^\dagger\otimes\mathbf{1})\sum_\beta d_\beta|\beta,\, \beta\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|k,\,k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\alpha,\,\alpha\rangle$$

- Consider n^2 operators (matrices) $\mathbf{a}[i\alpha],\ i,\alpha = 1,\cdots,n = \dim\mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$S_{\Psi|\Phi}\left((\mathbf{a}[i\alpha]\otimes\mathbf{1})\sum_k c_k\,|k,\,k\rangle\right) = (\mathbf{a}[i\alpha]^\dagger\otimes\mathbf{1})\sum_\beta d_\beta\,|\beta,\,\beta\rangle$$

$$S_{\Psi|\Phi}\left(c_i\,|\alpha,\,i\rangle\right) = d_\alpha\,|i,\,\alpha\rangle \ \Rightarrow\ \bar{c}_i\ S_{\Psi|\Phi}\,|\alpha,\,i\rangle = d_\alpha\,|i,\,\alpha\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|k,\,k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\alpha,\,\alpha\rangle$$

- Consider n^2 operators (matrices) $\mathbf{a}[i\alpha],\ i,\alpha = 1,\cdots,n = \dim\mathscr{H}_1$ as a basis of the algebra $\mathfrak{A}$,

$$S_{\Psi|\Phi}\left((\mathbf{a}[i\alpha]\otimes\mathbf{1})\sum_k c_k\,|k,\,k\rangle\right) = (\mathbf{a}[i\alpha]^\dagger\otimes\mathbf{1})\sum_\beta d_\beta\,|\beta,\,\beta\rangle$$

$$S_{\Psi|\Phi}\left(c_i\,|\alpha,\,i\rangle\right) = d_\alpha\,|i,\,\alpha\rangle \;\Rightarrow\; \bar{c}_i\,S_{\Psi|\Phi}\,|\alpha,\,i\rangle = d_\alpha\,|i,\,\alpha\rangle$$

$$S_{\Psi|\Phi}\,|\alpha,\,i\rangle = \frac{d_\alpha}{\bar{c}_i}\,|i,\,\alpha\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Phi, \quad \Psi = \sum_k c_k |k, k\rangle, \quad \Phi = \sum_\alpha d_\alpha |\alpha, \alpha\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|k,\,k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\alpha,\,\alpha\rangle$$

$$\langle\beta,\,l\,|\left(S^\dagger_{\Psi|\Phi}\,|\,i,\,\alpha\rangle\right) = \langle i,\,\alpha\,|\,(S_{\Psi|\Phi}\,|\,\beta,\,l\rangle) = \langle i,\,\alpha\,|\left(\frac{d_\beta}{\bar{c}_l}\,|\,l,\,\beta\rangle\right) = \frac{d_\beta}{\bar{c}_l}\langle i,\,\alpha\,|\,l,\,\beta\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|\,k,\,k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\,\alpha,\,\alpha\rangle$$

$$\langle\beta,\,l\,|\left(S^\dagger_{\Psi|\Phi}\,|\,i,\,\alpha\rangle\right) = \langle i,\,\alpha\,|\,(S_{\Psi|\Phi}\,|\,\beta,\,l\rangle) = \langle i,\,\alpha\,|\left(\frac{d_\beta}{\bar{c}_l}\,|\,l,\,\beta\rangle\right) = \frac{d_\beta}{\bar{c}_l}\langle i,\,\alpha\,|\,l,\,\beta\rangle$$

$$\Rightarrow \quad S^\dagger_{\Psi|\Phi}\,|\,i,\,\alpha\rangle = \frac{d_\alpha}{\bar{c}_i}\,|\,\alpha,\,i\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|\,k,\,k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\,\alpha,\,\alpha\rangle$$

$$\langle\beta,\,l\,|\left(S^\dagger_{\Psi|\Phi}\,|\,i,\,\alpha\rangle\right) = \langle i,\,\alpha\,|\,(S_{\Psi|\Phi}\,|\,\beta,\,l\rangle) = \langle i,\,\alpha\,|\left(\frac{d_\beta}{\bar{c}_l}\,|\,l,\,\beta\rangle\right) = \frac{d_\beta}{\bar{c}_l}\langle i,\,\alpha\,|\,l,\,\beta\rangle$$

$$\Rightarrow\quad S^\dagger_{\Psi|\Phi}\,|\,i,\,\alpha\rangle = \frac{d_\alpha}{\bar{c}_i}\,|\,\alpha,\,i\rangle$$

$$\Delta_{\Psi|\Phi}\,|\,\alpha,\,i\rangle = S^\dagger_{\Psi|\Phi}S_{\Psi|\Phi}\,|\,\alpha,\,i\rangle = S^\dagger_{\Psi}\left(\frac{d_\alpha}{\bar{c}_i}\,|\,i,\,\alpha\rangle\right) = \frac{\bar{d}_\alpha}{c_i}S^\dagger_{\Psi|\Phi}\,|\,i,\,\alpha\rangle$$

$$= \frac{\bar{d}_\alpha d_\alpha}{c_i\bar{c}_i}\,|\,\alpha,\,i\rangle = \frac{|\,d_\alpha\,|^2}{|\,c_i\,|^2}\,|\,\alpha,\,i\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a} \otimes \mathbf{1})\Psi) = (\mathbf{a}^\dagger \otimes \mathbf{1})\Phi, \quad \Psi = \sum_k c_k |k, k\rangle, \quad \Phi = \sum_\alpha d_\alpha |\alpha, \alpha\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^\dagger\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|\,k,\,k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\,\alpha,\,\alpha\rangle$$

$$\frac{d_\alpha}{\bar{c}_i}\,|\,i,\,\alpha\rangle = S_{\Psi|\Phi}\,|\,\alpha,\,i\rangle = J_{\Psi|\Phi}\Delta^{1/2}_{\Psi|\Phi}\,|\,\alpha,\,i\rangle = \sqrt{\frac{|\,d_\alpha\,|^2}{|\,c_i\,|^2}}\,J_{\Psi|\Phi}\,|\,\alpha,\,i\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}((\mathbf{a}\otimes\mathbf{1})\Psi) = (\mathbf{a}^{\dagger}\otimes\mathbf{1})\Phi, \quad \Psi = \sum_k c_k\,|k,\,k\rangle, \quad \Phi = \sum_\alpha d_\alpha\,|\alpha,\,\alpha\rangle$$

$$\frac{d_\alpha}{\bar{c}_i}\,|i,\,\alpha\rangle = S_{\Psi|\Phi}\,|\alpha,\,i\rangle = J_{\Psi|\Phi}\Delta^{1/2}_{\Psi|\Phi}\,|\alpha,\,i\rangle = \sqrt{\frac{|d_\alpha|^2}{|c_i|^2}}\,J_{\Psi|\Phi}\,|\alpha,\,i\rangle$$

$$\therefore\;\; J_{\Psi|\Phi}\,|\alpha,\,i\rangle = \frac{d_\alpha}{\bar{c}_i}\sqrt{\frac{c_i\bar{c}_i}{d_\alpha\bar{d}_\alpha}}\,|i,\,\alpha\rangle = \sqrt{\frac{c_i d_\alpha}{\bar{c}_i\bar{d}_\alpha}}\,|i,\,\alpha\rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The relative operators $S_{\Psi|\Phi}$, $\Delta_{\Psi|\Phi}$ and $J_{\Psi|\Phi}$

$$S_{\Psi|\Phi}|\alpha, i\rangle = \frac{d_\alpha}{\bar{c}_i}|i, \alpha\rangle$$

$$\Delta_{\Psi|\Phi}|\alpha, i\rangle = \frac{|d_\alpha|^2}{|c_i|^2}|\alpha, i\rangle, \qquad J_{\Psi|\Phi}|\alpha, i\rangle = \sqrt{\frac{c_i d_\alpha}{\bar{c}_i \bar{d}_\alpha}}|i, \alpha\rangle$$

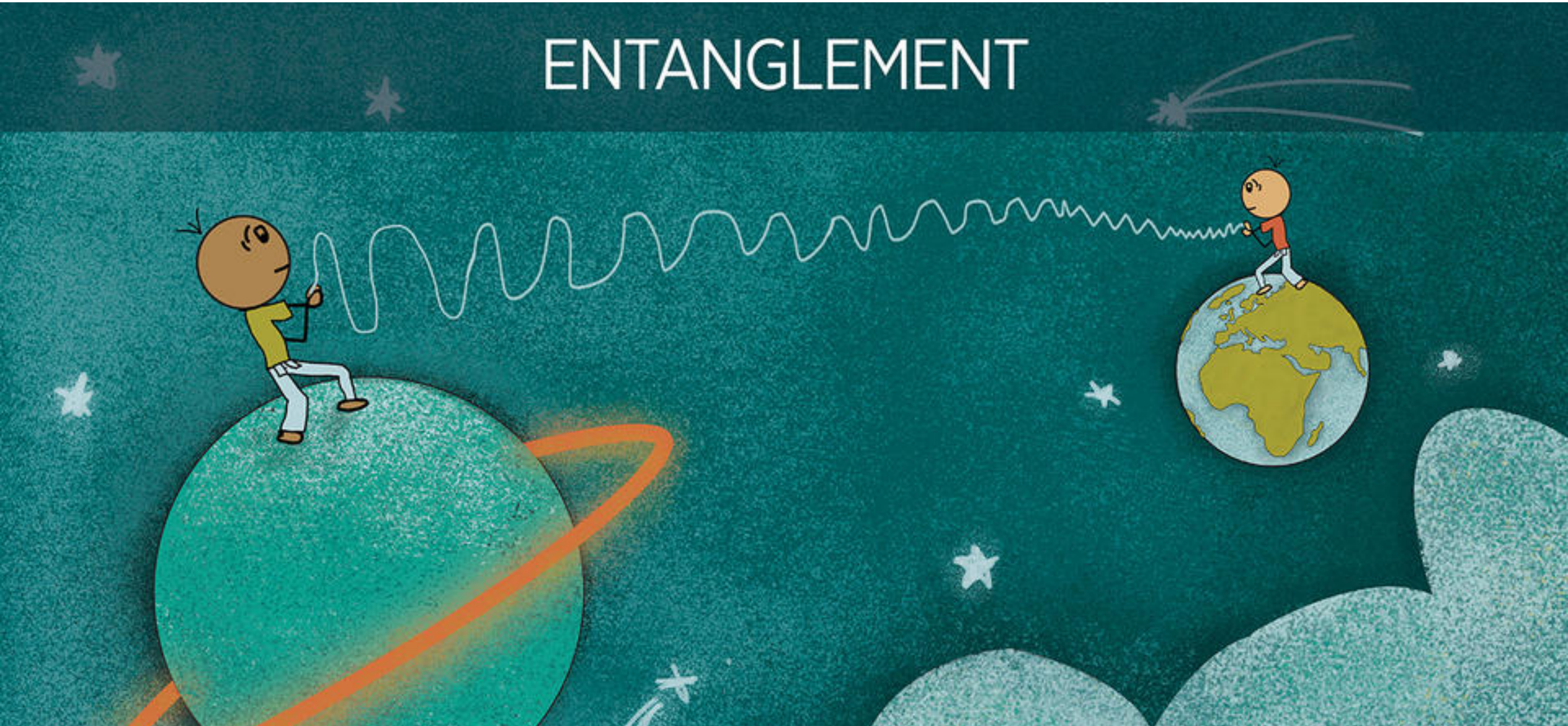
- One can always pick the phases of $|i\rangle$ and $|\alpha\rangle$ to ensure that the c_i and the d_α are all positive.

- In such a choice of the phases, $J_{\Psi|\Phi}|\alpha, i\rangle = |i, \alpha\rangle$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)

- For normalized states $\Psi, \Phi \in \mathscr{H}_1 \otimes \mathscr{H}_2$,

$$\sum_i |c_i|^2 = 1, \quad \sum_\alpha |d_\alpha|^2 = 1$$

- The density matrix $\hat{\rho}_{12} = |\Psi\rangle\langle\Psi|$ is a projective operator (to the 1-dim subspace generated by Ψ) on $\mathscr{H}$

$$\mathbf{Tr}_{12}\hat{\rho}_{12} = 1, \;\; \hat{\rho}_{12}^2 = \hat{\rho}_{12}, \;\; \mathbf{rank}\ \hat{\rho}_{12} = 1$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)

- For normalized states $\Psi, \Phi \in \mathscr{H}_1 \otimes \mathscr{H}_2$,

$$\sum_i |c_i|^2 = 1, \qquad \sum_\alpha |d_\alpha|^2 = 1$$

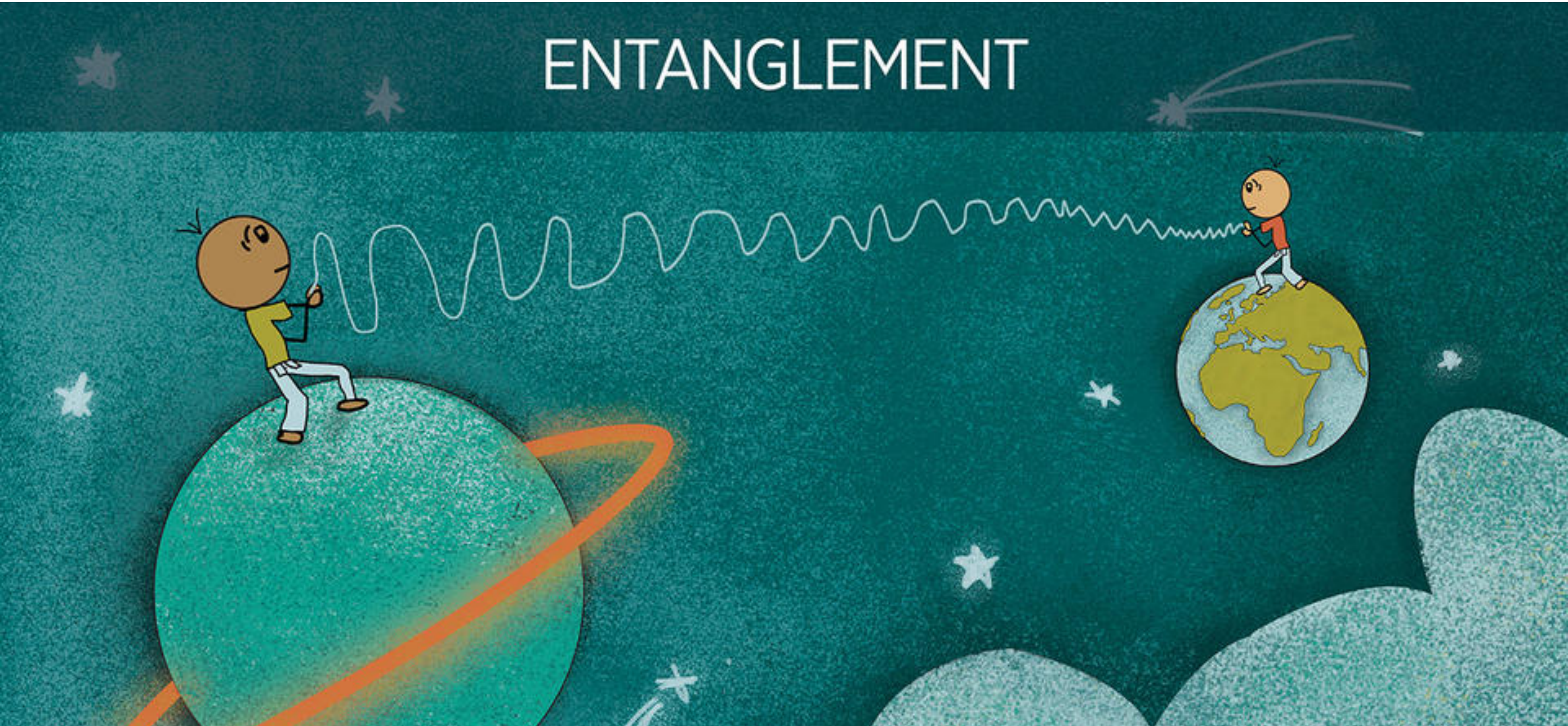
- The density matrix $\hat{\rho}_{12} = |\Psi\rangle\langle\Psi|$ is a projective operator (to the 1-dim subspace generated by Ψ) on $\mathscr{H}$

$$\mathbf{Tr}_{12}\hat{\rho}_{12} = 1, \;\; \hat{\rho}_{12}^2 = \hat{\rho}_{12}, \;\; \mathbf{rank}\ \hat{\rho}_{12} = 1$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)



FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Partial trace over $\mathscr{H}_1$ or $\mathscr{H}_2$:

$$\mathbf{Tr}_1 \hat{\mathcal{O}} = \sum_i \langle i, \ \cdot \,|\, \hat{\mathcal{O}} \,|\, i, \cdot \ \rangle$$

- For example, the reduced density matrix

$$\hat{\rho}_1 \equiv \mathbf{Tr}_2 \hat{\rho}_{12} = \sum_j \langle \ \cdot \ , \, j \,|\, \hat{\rho}_{12} \,|\, \cdot \ , \, j \rangle$$

$$= \sum_{j,i} |\, c_i \,|^2 \langle \ \cdot \ , \, j \,|\, i, \, i \rangle \langle i, \, i \,|\, \cdot \ , \, j \rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Partial trace over $\mathscr{H}_1$ or $\mathscr{H}_2$:

$$\mathbf{Tr}_1 \hat{\mathcal{O}} = \sum_i \langle i, \ \cdot \,|\, \hat{\mathcal{O}} \,|\, i, \cdot \ \rangle$$

- For example, the reduced density matrix

$$\hat{\rho}_1 \equiv \mathbf{Tr}_2 \hat{\rho}_{12} = \sum_j \langle \ \cdot \ , \ j \,|\, \hat{\rho}_{12} \,|\, \cdot \ , \ j \rangle$$

$$= \sum_{j,i} |\, c_i \,|^2 \langle \ \cdot \ , \ j \,|\, i, \ i \rangle \langle i, \ i \,|\, \cdot \ , \ j \rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)

- Partial trace over $\mathscr{H}_1$ or $\mathscr{H}_2$:

$$\mathbf{Tr}_1 \hat{\mathcal{O}} = \sum_i \langle i, \ \cdot \, | \hat{\mathcal{O}} | i, \cdot \, \rangle$$

- For example, the reduced density matrix

$$\hat{\rho}_1 \equiv \mathbf{Tr}_2 \hat{\rho}_{12} = \sum_j \langle \, \cdot \, , j | \hat{\rho}_{12} | \, \cdot \, , j \rangle$$

$$= \sum_{j,i} |c_i|^2 \langle \, \cdot \, , j | i, i \rangle \langle i, i | \, \cdot \, , j \rangle$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Partial trace over $\mathscr{H}_1$ or $\mathscr{H}_2$:

$$\mathbf{Tr}_1 \hat{\mathcal{O}} = \sum_i \langle i, \ \cdot \,|\, \hat{\mathcal{O}} \,|\, i, \cdot \, \rangle$$

- For example, the reduced density matrix

$$\hat{\rho}_1 \equiv \mathbf{Tr}_2 \hat{\rho}_{12} = \sum_j \langle \, \cdot \, , \, j \,|\, \hat{\rho}_{12} \,|\, \cdot \, , \, j \rangle$$

$$= \sum_{j,i} |\, c_i \,|^2 \langle \, \cdot \, , \, j \,|\, i, \, i \rangle \langle i, \, i \,|\, \cdot \, , \, j \rangle = \sum_i |\, c_i \,|^2 \,|\, i \rangle \langle i \,|$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Partial trace over $\mathscr{H}_1$ or $\mathscr{H}_2$:

$$\mathbf{Tr}_1 \hat{\mathcal{O}} = \sum_i \langle i, \ \cdot \,|\, \hat{\mathcal{O}} \,|\, i, \cdot \,\rangle$$

- For example, the reduced density matrices

$$\hat{\rho}_1 = \sum_i |c_i|^2 |\psi_i\rangle\langle\psi_i|, \quad \hat{\rho}_2 = \sum_i |c_i|^2 |\varphi_i\rangle\langle\varphi_i|$$

- They are invertible iff the c_i are all nonzero.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Partial trace over $\mathscr{H}_1$ or $\mathscr{H}_2$:

$$\mathbf{Tr}_1 \hat{\mathscr{O}} = \sum_i \langle i, \ \cdot \,|\, \hat{\mathscr{O}} \,|\, i, \cdot \,\rangle$$

- The reduced density matrices for Ψ and Φ:

$$\hat{\rho}_1 = \sum_i |c_i|^2 |\psi_i\rangle\langle\psi_i|, \quad \hat{\rho}_2 = \sum_i |c_i|^2 |\varphi_i\rangle\langle\varphi_i|$$

$$\hat{\sigma}_1 = \sum_\alpha |d_\alpha|^2 |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha|, \quad \hat{\sigma}_2 = \sum_\alpha |d_\alpha|^2 |\tilde{\varphi}_\alpha\rangle\langle\tilde{\varphi}_\alpha|$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Rewriting the modular operator with reduced density matrices:

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Rewriting the modular operator with reduced density matrices:

$$\Delta_\Psi = \Delta_\Psi \sum_{i,j} |i,j\rangle\langle i,j| = \sum_{i,j} \frac{|c_i|^2}{|c_j|^2} |i,j\rangle\langle i,j| = \sum_{i,j} \frac{|c_i|^2}{|c_j|^2} |\psi_i\rangle\langle\psi_i| \otimes |\varphi_j\rangle\langle\varphi_j|$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Rewriting the modular operator with reduced density matrices:

$$\Delta_\Psi = \Delta_\Psi \sum_{i,j} |i,\, j\rangle\langle i,\, j| = \sum_{i,j} \frac{|c_i|^2}{|c_j|^2} |i,\, j\rangle\langle i,\, j| = \sum_{i,j} \frac{|c_i|^2}{|c_j|^2} |\psi_i\rangle\langle\psi_i| \otimes |\varphi_j\rangle\langle\varphi_j|$$

$$= \sum_{i,j} (|c_i|^2 |\psi_i\rangle\langle\psi_i|) \otimes (|c_j|^{-2} |\varphi_j\rangle\langle\varphi_j|) = \left(\sum_i |c_i|^2 |\psi_i\rangle\langle\psi_i|\right)\left(\sum_j |c_j|^{-2} |\varphi_j\rangle\langle\varphi_j|\right)$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Rewriting the modular operator with reduced density matrices:

$$\Delta_\Psi = \Delta_\Psi \sum_{i,j} |i,\, j\rangle\langle i,\, j| = \sum_{i,j} \frac{|c_i|^2}{|c_j|^2} |i,\, j\rangle\langle i,\, j| = \sum_{i,j} \frac{|c_i|^2}{|c_j|^2} |\psi_i\rangle\langle\psi_i| \otimes |\varphi_j\rangle\langle\varphi_j|$$

$$= \sum_{i,j} (|c_i|^2 |\psi_i\rangle\langle\psi_i|) \otimes (|c_j|^{-2} |\varphi_j\rangle\langle\varphi_j|) = \left(\sum_i |c_i|^2 |\psi_i\rangle\langle\psi_i|\right)\left(\sum_j |c_j|^{-2} |\varphi_j\rangle\langle\varphi_j|\right)$$

$$= \hat{\rho}_1 \otimes \hat{\rho}_2^{-1}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- From pure states to mixed states (inverse of purification)
- Rewriting the relative modular operator with reduced density matrices:

$$\Delta_{\Psi|\Phi} = \Delta_{\Psi|\Phi} \sum_{\alpha,i} |\alpha, i\rangle\langle\alpha, i| = \sum_{\alpha,i} \frac{|d_\alpha|^2}{|c_i|^2} |\alpha, i\rangle\langle\alpha, i| = \sum_{\alpha,i} \frac{|d_\alpha|^2}{|c_i|^2} |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha| \otimes |\varphi_i\rangle\langle\varphi_i|$$

$$= \sum_{\alpha,i} (|d_\alpha|^2 |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha|) \otimes (|c_i|^{-2} |\varphi_i\rangle\langle\varphi_i|) = \left(\sum_\alpha |d_\alpha|^2 |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha| \right) \left(\sum_i |c_i|^{-2} |\varphi_i\rangle\langle\varphi_i| \right)$$

$$= \hat{\sigma}_1 \otimes \hat{\rho}_2^{-1}$$

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The “representation matrices” of modular operators

- The cyclic and separating vector Ψ and the induced antiunitary modular conjugation $J_\Psi |i, j\rangle = |j, i\rangle$ gives a special linear bijective from $\mathscr{H}_1$ to $\overline{\mathscr{H}_2}$, so they identifies $\mathscr{H}_2$ with the dual of the $\mathscr{H}_1$.

FINITE-DIMENSIONAL QUANTUM SYSTEMS AND SOME LESSONS

I. The modular operators in the finite-dimensional case

- The "representation matrices" of modular operators
- The antiunitary modular conjugation

$$J_\Psi |i, j\rangle = |j, i\rangle$$

$$\Xi = \sum_{i,j=1}^{n} c_{ij} |i, j\rangle = \mathrm{tr}\left[\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}\begin{pmatrix} |1, 1\rangle & |2, 1\rangle & \cdots & |n, 1\rangle \\ |1, 2\rangle & |2, 2\rangle & \cdots & |n, 2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |1, n\rangle & |2, n\rangle & \cdots & |n, n\rangle \end{pmatrix}\right]$$

$$J_\Psi \Xi = \sum_{i,j=1}^{n} \bar{c}_{ij} |j, i\rangle = \mathrm{tr}\left[\begin{pmatrix} \bar{c}_{11} & \bar{c}_{21} & \cdots & \bar{c}_{n1} \\ \bar{c}_{12} & \bar{c}_{22} & \cdots & \bar{c}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}_{1n} & \bar{c}_{2n} & \cdots & \bar{c}_{nn} \end{pmatrix}\begin{pmatrix} |1, 1\rangle & |2, 1\rangle & \cdots & |n, 1\rangle \\ |1, 2\rangle & |2, 2\rangle & \cdots & |n, 2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |1, n\rangle & |2, n\rangle & \cdots & |n, n\rangle \end{pmatrix}\right]$$


To Be Continued...